

GENERAL DYNAMIC TERM STRUCTURES UNDER DEFAULT RISK

CLAUDIO FONTANA AND THORSTEN SCHMIDT

ABSTRACT. We consider the problem of modelling the term structure of bonds subject to default risk, under minimal assumptions on the default time. In particular, we do not assume the existence of a default intensity and we therefore allow for the possibility of default at predictable times. It turns out that this requires the introduction of an additional term to the forward-rate approach by Heath, Jarrow and Morton (1992). This term is driven by a random measure encoding information about those times where default can happen with positive probability. In this framework, we derive necessary and sufficient conditions for a reference probability measure to be a local martingale measure for the large financial market of credit risky bonds, also considering general recovery schemes. To this end, we establish a new Fubini theorem with respect to a random measure by means of enlargement of filtrations techniques.

1. INTRODUCTION

The study of the evolution of the term structure of bond prices in the presence of default risk typically starts from a forward rate model similar to the classical approach of Heath, Jarrow and Morton (HJM) in [22]. In this approach, bond prices are assumed to be absolutely continuous with respect to the lifetime of the bond (maturity). This assumption is typically justified by the argument that, in practice, only a finite number of bonds are liquidly traded and the full term structure is obtained by interpolation, hence is smooth.

In markets with default risk, however, discontinuities are rather the rule than the exception: the seminal work of Merton [32] clearly shows such a behavior, as do many other *structural models* (see, e.g., [2, 19, 20]). A default event usually occurs in correspondence of a missed payment by a corporate or sovereign entity and, in many cases, the payment dates are publicly known in advance. The missed coupon payments by Argentina on a notional of \$29 billion (on July 30, 2014; see [23]) and by Greece on a notional of €1.5 billion (on June 30, 2015; see [11]) are prime examples of credit events occurring at predetermined payment dates. It is therefore natural to expect the term structure of credit risky bonds to exhibit discontinuities in correspondence of such payment dates.¹

On the other side, *reduced-form models* (see [1, 9, 16, 25, 31] for some of the first works in this direction) are less ambitious about the precise mechanism leading to default and neglect this phenomenon. Reduced-form models generally assume the existence of a *default intensity*, thus implying that the probability of the default event occurring at any predictable time vanishes. Accordingly, reduced-form HJM-type models for defaultable term structures typically postulate that, prior to default, bond prices are absolutely continuous with respect to maturity, i.e., under

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¹As an illustrative example, the timeline of the payment dates on Greece's debt is publicly available and daily updated at <http://graphics.wsj.com/greece-debt-timeline>.

the assumption of zero recovery, credit risky bond prices $P(t, T)$ are described by

$$(1.1) \quad P(t, T) = \mathbb{1}_{\{\tau > t\}} \exp \left(- \int_t^T f(t, u) du \right),$$

with τ denoting the random default time and $(f(t, T))_{0 \leq t \leq T}$ the instantaneous forward rate. This approach has been studied in numerous works and up to a great level of generality, beginning with the first works [10, 27, 37, 38] and extended in various directions in [12, 13, 33, 36] (see [3, Chapter 13] for an overview of the relevant literature).

It turns out that, assuming absence of arbitrage, the presence of predictable times at which the default event can occur with strictly positive probability is incompatible with an absolutely continuous term structure of the form (1.1). This fact, already pointed out in 1998 in [38], has motivated more general approaches such as [2] and [18]. More specifically, in the recent paper [18], the classical reduced-form HJM approach is extended by allowing the *default compensator* (i.e., the compensator of the increasing process $\mathbb{1}_{\llbracket \tau, +\infty \rrbracket}$) to have an absolutely continuous part, corresponding to a default intensity, as well as a discrete part with a finite number of jumps. In particular, the presence of jumps allows that the default event can occur with strictly positive probability at predictable times, which are supposed to be known in advance in the market. In this context, in order to exclude arbitrage possibilities, the term structure equation (1.1) has to be extended by introducing discontinuities in correspondence to those times.

In the present paper, we introduce a general framework for the modelling of defaultable term structures under minimal assumptions, going significantly beyond the intensity-based approach and generalizing the approach of [18]. More specifically, we refrain from making any assumption on the random default time τ as well as on the default compensator, allowing in particular for the default event to occur with strictly positive probability at predictable times. A natural and general way to represent the term structure of credit risky bonds, also allowing for discontinuities, is to extend (1.1) to the following specification:

$$(1.2) \quad P(t, T) = \mathbb{1}_{\{\tau > t\}} \exp \left(- \int_t^T f(t, u) du - \int_{(t, T]} g(t, u) \mu_t(du) \right),$$

where $(\mu_t(du))_{t \geq 0}$ is a measure-valued process with possibly singular and jump parts and where $(f(t, T))_{t \geq 0}$ and $((g(t, T)))_{t \geq 0}$ are two random fields representing instantaneous forward rates. The additional term $\int_{(t, T]} g(t, u) \mu_t(du)$ can be interpreted as the effect of the information received up to date t about possible “risky dates” (i.e., periods at which the default event can occur with strictly positive probability) in the remaining lifetime $(t, T]$ of the bond.

In this general setting, we obtain necessary and sufficient conditions for a reference probability measure \mathbb{Q} to be a local martingale measure for the infinite-dimensional financial market composed by all credit risky bonds with prices given by (1.2), thereby ensuring absence of arbitrage in a sense to be precisely specified below. To this end, we provide a new Fubini theorem with respect to random measures by combining techniques of enlargement of filtrations and stochastic integration depending on a parameter. Furthermore, we also study the extension of (1.2) to the case of a general recovery process over multiple default dates.

In overall terms, the present paper can be regarded as a general HJM-type framework bridging the gap between intensity-based and structural models. Moreover, despite the level of generality, our HJM-type conditions admit a clear economic interpretation and can be further simplified in

several special cases of practical interest, notably in the case where the process $(\mu_t(du))_{t \geq 0}$ is generated by an integer-valued random measure.

The paper is structured as follows. Section 2 contains a description of the setting and the main technical assumptions and presents a general decomposition of the default compensator process. The main results of the paper are presented in Section 3, first in the case of zero recovery at default and then for a general recovery process. Special cases and examples will also be discussed, together with relations to the literature. Section 4 contains the proofs of all our results, while Section 5 provides a new stochastic Fubini theorem with respect to a random measure.

2. GENERAL DEFAULTABLE TERM STRUCTURE MODELS

2.1. Setting. Let $(\Omega, \mathcal{A}, \mathbb{Q})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions (i.e., \mathbb{F} is right-continuous and, if $A \subseteq B \in \mathcal{A}$ and $\mathbb{Q}(B) = 0$, then $A \in \mathcal{F}_0$), with $T < +\infty$ denoting a fixed time horizon. We assume that the filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{Q})$ is sufficiently rich to support an n -dimensional Brownian motion $W = (W_t)_{0 \leq t \leq T}$ and an optional non-negative random measure $\mu(ds, du)$ on $[0, T] \times [0, T]$. Throughout the paper, the probability measure \mathbb{Q} will represent a reference probability measure. We follow the notation of [24] and refer to this work for details on stochastic processes which are not laid out here.

2.2. The default time. We consider an abstract financial market containing an entity (e.g., a company) which may default at the random *default time* τ . The filtration \mathbb{F} is meant to represent all information publicly available in the market and, therefore, we assume that the random time τ is an \mathbb{F} -stopping time (i.e., the default event is publicly observable). We define the associated *default indicator process* $H = (H_t)_{0 \leq t \leq T}$ by $H_t := \mathbb{1}_{\{\tau \leq t\}}$, for $t \in [0, T]$. We will also make use of the *survival process* $1 - H$. The process $H = \mathbb{1}_{[\tau, T]}$ is \mathbb{F} -adapted, bounded, increasing and right-continuous. As such, by the Doob-Meyer decomposition (see, e.g., [24, Theorem I.3.15]), there exists a unique predictable, integrable and increasing process $H^p = (H_t^p)_{0 \leq t \leq T}$ with $H_0^p = 0$, called the *default compensator* (or dual predictable projection of H), such that the process $H - H^p$ is a uniformly integrable martingale on $(\Omega, \mathbb{F}, \mathbb{Q})$. Note also that $H_t^p = H_{\tau \wedge t}^p$, for all $t \in [0, T]$.

Apart from the minimal assumption of being an \mathbb{F} -stopping time, we do not introduce any further assumption on the default time τ . Hence, in this general framework, the default compensator H^p is not necessarily absolutely continuous (i.e., a *default intensity* may fail to exist) and may also contain both singular and jump parts, as shown in the following lemma (all proofs will be given in Section 4).

Lemma 2.1. *The default compensator H^p admits a unique decomposition*

$$(2.1) \quad H_t^p = \int_0^t h_s ds + \lambda_t + \sum_{0 < s \leq t} \Delta H_s^p, \quad \text{for all } 0 \leq t \leq T,$$

where $(h_t)_{0 \leq t \leq T}$ is a non-negative predictable process such that $\int_0^T |h_s| ds < +\infty$ a.s. and $(\lambda_t)_{0 \leq t \leq T}$ is an increasing and continuous process with $\lambda_0 = 0$ such that $d\lambda_s(\omega) \perp ds$, for a.a. $\omega \in \Omega$.

We denote by $\bigcup_{i \in \mathbb{N}} \llbracket U_i \rrbracket$ the thin set of the random jump times of the default compensator H^p , where $\{U_i\}_{i \in \mathbb{N}}$ is a family of predictable times. By [21, Theorem 5.27], it holds that

$$\mathbb{Q}(\tau = U_i \leq T) = \mathbb{E}[\Delta H_{U_i}] = \mathbb{E}[\Delta H_{U_i}^p] > 0, \quad \text{for all } i \in \mathbb{N},$$

meaning that the default event has a strictly positive probability of occurrence in correspondence of the predictable dates $\{U_i\}_{i \in \mathbb{N}}$. The classical *intensity-based approach* is obtained as a special case by letting $\lambda = \Delta H^p = 0$ in decomposition (2.1).

2.3. The term structure of credit risky bonds. A *credit risky bond* with maturity $T \in [0, \mathbb{T}]$ is a contingent claim promising to pay one unit of currency at maturity T , provided that the defaultable entity does not default before date T . We denote by $P(t, T)$ the price at date t of a credit risky bond with maturity T , for all $0 \leq t \leq T \leq \mathbb{T}$. As a first step, we restrict our attention to the *zero-recovery* case, meaning that we assume that the credit risky bond becomes worthless as soon as the default event occurs, i.e., $P(t, T) = 0$ if $H_t = 1$, for all $0 \leq t \leq T \leq \mathbb{T}$ (see Section 3.4 for the analysis of general recovery schemes).

The family of stochastic processes $\{(P(t, T)_{0 \leq t \leq T}); T \in [0, \mathbb{T}]\}$ describes the evolution of the *term structure* $T \mapsto P(\cdot, T)$ over time. Following the extended HJM-framework suggested in [18], we assume that the term structure of credit risky bonds satisfies

$$(2.2) \quad P(t, T) = (1 - H_t) \exp \left(- \int_t^T f(t, u) du - \int_{(t, T]} g(t, u) \mu_t(du) \right), \quad \text{for all } 0 \leq t \leq T \leq \mathbb{T};$$

here $\mu(du) = (\mu_t(du))_{0 \leq t \leq \mathbb{T}}$ is the measure-valued process defined by $\mu_t(du) := \mu([0, t] \times du)$, for $t \in [0, \mathbb{T}]$, with $\mu(ds, du)$ being the random measure introduced in Section 2.1. The processes f and g are assumed to be of the form²

$$(2.3) \quad f(t, T) = f(0, T) + \int_0^t a(s, T) ds + \int_0^t b(s, T) dW_s,$$

$$(2.4) \quad g(t, T) = g(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \beta(s, T) dW_s,$$

for all $0 \leq t \leq T \leq \mathbb{T}$. The precise technical assumptions on the random measure μ as well as on the processes appearing in (2.3)-(2.4) will be given in Section 2.4 below. For the moment, let us briefly comment on the interpretation of the term structure equation (2.2).

Remark 2.2. In comparison with the classical HJM framework applied to credit risk for example in [10, 27, 37, 38], the novelty of the term structure equation (2.2) consists in the presence of the random measure μ . This random measure encodes the information received over time about possible “risky dates” or “risky periods” where, on the basis of the available information, the default event is perceived to be more likely to happen (explicit examples will be considered in Section 3.3). More specifically, the first argument of μ measures is as usual the running time, while the second argument of μ identifies the possible risky dates and periods. Hence, the integral with respect to $\mu_t(du)$ appearing in (2.2) represents all the information received up to date t concerning possible risky dates and on the likelihood of default in the remaining lifetime $(t, T]$ of the bond. The assumption that μ is an optional random measure simply captures the fact that this information about future risky dates is publicly available, but may suddenly arrive in the market (since μ is not necessarily predictable). Moreover, it could also happen that future periods which are perceived to be risky at some date t_1 are no longer considered risky at a later date $t_2 > t_1$ (and

²We want to point out that our results can be extended to the case where f and g are more general semimartingale random fields. However, since our main goal consists in studying defaultable term structures driven by general random measures, we prefer to let f and g be of the simple form (2.3)-(2.4), in order not to obscure the presentation with painstaking technical aspects.

vice versa). As will be shown below, absence of arbitrage will imply a precise relationship between the default compensator H^p and the random measure μ .

2.4. Technical assumptions and preliminaries. We now formulate the technical assumptions needed for the analysis of the term structure introduced above, starting by making precise the assumptions on the random measure μ . As a preliminary, let us define the auxiliary filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \in [0, \mathbb{T}]}$ given by the *initial enlargement* of \mathbb{F} with respect to μ , i.e., $\tilde{\mathcal{F}}_t := \bigcap_{s > t} \tilde{\mathcal{F}}_s^0$, where

$$\tilde{\mathcal{F}}_t^0 := \mathcal{F}_t \vee \sigma\{\mu_t(A); t \in [0, \mathbb{T}], A \in \mathcal{B}([0, \mathbb{T}])\}, \quad \text{for all } 0 \leq t \leq \mathbb{T}.$$

The random measure μ is assumed to satisfy the following standing assumption.

Assumption 2.3. The random measure $\mu(ds, du)$ is a non-negative optional random measure on $[0, \mathbb{T}] \times [0, \mathbb{T}]$ satisfying the following properties:

- (i) $\mu(\omega; ds, du) = \mathbb{1}_{\{s < u\}} \mu(\omega; ds, du)$, for all $(s, u) \in [0, \mathbb{T}] \times [0, \mathbb{T}]$ and $\omega \in \Omega$;
- (ii) there exists a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of stopping times increasing a.s. to infinity such that $\mathbb{E}[\mu_{\sigma_n}([0, \mathbb{T}])] < +\infty$, for every $n \in \mathbb{N}$;
- (iii) the Brownian motion W is a semimartingale in the enlarged filtration $\tilde{\mathbb{F}}$.

According to the interpretation introduced in Remark 2.2, part (i) of Assumption 2.3 represents the fact that the new information received at date s only concerns the likelihood of default in the future (and not in the past). In view of (2.2), this assumption comes without loss of generality. Part (ii) is an integrability assumption, which ensures that the random measure μ is predictably σ -finite, in the sense of [24, Definition II.1.6], and that the random variable $\mu_t([0, \mathbb{T}])$ is a.s. finite, for all $t \in [0, \mathbb{T}]$.

Part (iii) of Assumption 2.3 is a joint hypothesis on the random measure μ and on the market filtration \mathbb{F} and, at first sight, looks as a technical assumption. It will be used to ensure the applicability of a suitable stochastic Fubini theorem (see Proposition 5.1 and the proof of Lemma 4.1). On a second view, it turns out that this assumption is indeed natural from a financial perspective. In fact, $\mu_t(du)$ encodes all available information at time t concerning possible risky dates/periods in the future. We do not assume that this information is independent of W (in which case the assumption clearly holds). However, we assume that adding this information to the market filtration at the initial date $t = 0$ may change the Brownian motion W to a semimartingale, but not beyond. In general, if W was not a semimartingale after this enlargement, then arbitrage possibilities would arise for an insider agent having a complete knowledge of μ at $t = 0$. However, this would represent a rather pathological situation, since the random measure μ does not encode actual information on the occurrence of the default event, but only information on future dates at which default is perceived to be possible with positive probability. In Section 3.3 we give some examples illustrating that this assumption is satisfied in several practically relevant cases.

Apart from Assumption 2.3, the random measure μ is allowed to be fully general. The following lemma presents a first consequence of Assumption 2.3. We define the process $\bar{\mu} = (\bar{\mu}_t)_{0 \leq t \leq \mathbb{T}}$ by

$$\bar{\mu}_t := \mu([0, \mathbb{T}] \times [0, t]), \quad \text{for all } t \in [0, \mathbb{T}].$$

Lemma 2.4. Suppose that parts (i)-(ii) of Assumption (2.3) hold. Then $\bar{\mu}$ is a predictable and increasing process, admitting the unique decomposition

$$(2.5) \quad \bar{\mu}_t = \int_0^t m_s ds + \nu_t + \sum_{0 < s \leq t} \Delta \bar{\mu}_s, \quad \text{for all } 0 \leq t \leq \mathbb{T},$$

where $(m_t)_{0 \leq t \leq \mathbb{T}}$ is a non-negative and predictable process satisfying $\int_0^{\mathbb{T}} |m_s| ds < +\infty$ a.s. and $(\nu_t)_{0 \leq t \leq \mathbb{T}}$ is an increasing and continuous process with $\nu_0 = 0$ such that $d\nu_s(\omega) \perp ds$, for almost all $\omega \in \Omega$.

The quantity $\bar{\mu}_t$ introduced in the above lemma measures the existence of risky dates in the period $[0, t]$, on the basis of all available information over $[0, \mathbb{T}]$, compare Remark 2.2. In a similar way, the quantity $\Delta \bar{\mu}_t = \mu([0, \mathbb{T}] \times \{t\})$ encodes whether the time t is perceived as a risky date, on the basis of all available information. As we shall see in Theorem 3.2, the absence of arbitrage implies a precise relationship between the terms appearing in the decompositions (2.1) and (2.5).

We will need the following mild technical assumptions, ensuring that all (stochastic) integrals are well-defined and that we can apply suitable versions of the (stochastic) Fubini theorem. We denote by \mathcal{O} (\mathcal{P} , resp.) the optional (predictable, resp.) σ -field on $(\Omega, \mathcal{A}, \mathbb{F})$.

Assumption 2.5. The following conditions hold a.s.:

- (i) The *initial forward curves* $T \mapsto f(\omega; 0, T)$ and $T \mapsto g(\omega; 0, T)$ are $\mathcal{F}_0 \otimes \mathcal{B}([0, \mathbb{T}])$ -measurable, real-valued, continuous and integrable on $[0, \mathbb{T}]$:

$$\int_0^{\mathbb{T}} |f(0, u)| du < +\infty \quad \text{and} \quad \int_0^{\mathbb{T}} |g(0, u)| \mu_t(du) < +\infty, \quad \text{for all } t \in [0, \mathbb{T}];$$

- (ii) the *drift processes* $a(\omega; s, u)$ and $\alpha(\omega; s, u)$ are $\mathcal{O} \otimes \mathcal{B}([0, \mathbb{T}])$ -measurable and real-valued, $a(\omega; s, u) = 0$ and $\alpha(\omega; s, u) = 0$ for all $0 \leq u < s \leq \mathbb{T}$, the maps $u \mapsto a(\omega; s, u)$ and $u \mapsto \alpha(\omega; s, u)$ are differentiable and satisfy

$$\begin{aligned} \int_0^{\mathbb{T}} \int_0^{\mathbb{T}} (|a(s, u)| + |\partial_u a(s, u)|) ds du &< +\infty, \\ \int_0^{\mathbb{T}} \int_0^{\mathbb{T}} |\partial_u \alpha(s, u)| ds du &< +\infty \quad \text{and} \quad \int_0^{\mathbb{T}} \int_0^{\mathbb{T}} |\alpha(s, u)| ds \mu_t(du) < +\infty, \quad \text{for all } t \in [0, \mathbb{T}]; \end{aligned}$$

- (iii) the *volatility processes* $b(\omega; s, u)$ and $\beta(\omega; s, u)$ are $\mathcal{O} \otimes \mathcal{B}([0, \mathbb{T}])$ -measurable and \mathbb{R}^n -valued, $b(\omega; s, u) = 0$ and $\beta(\omega; s, u) = 0$ for all $0 \leq u < s \leq \mathbb{T}$, the maps $u \mapsto b(\omega; s, u)$ and $u \mapsto \beta(\omega; s, u)$ are differentiable and satisfy

$$\begin{aligned} \int_0^{\mathbb{T}} \int_0^{\mathbb{T}} (\|b(s, u)\| + \|\partial_u b(s, u)\|)^2 ds du &< +\infty \\ \int_0^{\mathbb{T}} \int_0^{\mathbb{T}} \|\partial_u \beta(s, u)\|^2 ds du &< +\infty \quad \text{and} \quad \left(\sqrt{\int_0^{\mathbb{T}} (\beta^i(s, u))^2 \mu_s(du)} \right)_{0 \leq s \leq \mathbb{T}} \in L(W^i; \tilde{\mathbb{F}}), \end{aligned}$$

for all $i = 1, \dots, n$, where $L(W^i; \tilde{\mathbb{F}})$ denotes the set of all $\tilde{\mathbb{F}}$ -optional processes which are integrable with respect to W^i viewed as a continuous semimartingale in the enlarged filtration $\tilde{\mathbb{F}}$ (by part (iii) of Assumption 2.3)³.

It is easy to check that Assumption 2.5 implies that both the ordinary and stochastic integrals appearing in (2.3)-(2.4) are well-defined. Moreover, arguing similarly as in the proof of [17, Proposition 6.1], the processes $(f(t, t))_{0 \leq t \leq \mathbb{T}}$ and $(g(t, t))_{0 \leq t \leq \mathbb{T}}$ are continuous, hence predictable and locally bounded. By an analogous argument, it can be shown that, for any fixed $t \in [0, \mathbb{T}]$

³The process $(\sqrt{\int_0^{\mathbb{T}} (\beta^i(s, u))^2 \mu_s(du)})_{0 \leq s \leq \mathbb{T}}$ is optional but not necessarily predictable. However, in view of [21, Theorem 3.20] and since W^i is a continuous $\tilde{\mathbb{F}}$ -semimartingale, this does not cause any problem for stochastic integration (compare also with [24, part 3 of Remark III.6.28]).

and for a.a. $\omega \in \Omega$, the maps $u \mapsto f(\omega; t, u)$ and $u \mapsto g(\omega; t, u)$ are continuous. In turn, together with part (ii) of Assumption 2.3, this implies that both integrals appearing in the term structure equation (2.2) are a.s. finite. Finally, note that $\int_0^\cdot g(s, s) d\bar{\mu}_s$ is well-defined as a predictable process of finite variation (see, e.g., [24, Proposition I.3.5]).

3. HJM-TYPE CONDITIONS FOR GENERAL DEFAULTABLE TERM STRUCTURES

This section contains our main results and characterizes the absence of arbitrage in the context of general term structure models as introduced in Section 2. After making precise in Section 3.1 the description of the financial market and the notion of arbitrage we consider, Section 3.2 presents the main theorem, while Section 3.3 deals with several special cases of interest and Section 3.4 presents the extension to general recovery schemes. Finally, we discuss related works in Section 3.5. The proofs of all the results will be given in Section 4.

3.1. The large financial market of credit risky bonds. The considered financial market is assumed to contain a *numéraire*, whose price process is strictly positive, càdlàg and adapted, and denoted by $X^0 = (X_t^0)_{0 \leq t \leq T}$. Without loss of generality, we assume that $X_0^0 = 1$. Moreover, we make the classical assumption that X^0 is absolutely continuous, i.e., there exists a predictable integrable short-rate process $r = (r_t)_{0 \leq t \leq T}$ such that $X_t^0 = \exp(\int_0^t r_s ds)$, for all $t \in [0, T]$. For practical applications, one would typically use the overnight index swap (OIS) rate for constructing the numéraire.

The credit risky bond market consists of the uncountable family $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, T]\}$, representing the price processes of all basic traded assets. In particular, this financial market is infinite-dimensional and, therefore, can be treated as a *large financial market*, in the spirit of [5, 30]. This corresponds to considering sequences of trading strategies, each strategy only consisting of portfolios of finitely many but arbitrary credit risky bonds, and taking the limits of those. If the limit is taken in Émery's semimartingale topology, this leads to the notion of *no asymptotic free lunch with vanishing risk (NAFLVR)* recently introduced in a general setting in [5]. In particular, NAFLVR holds in our context if the probability measure \mathbb{Q} is a *local martingale measure* for the family $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, T]\}$ with respect to the numéraire X^0 , i.e., if the process $P(\cdot, T)/X^0$ is a \mathbb{Q} -local martingale, for every $T \in [0, T]$. In the following, we will derive necessary and sufficient conditions for this property to hold, thus ensuring that the credit risky bond market is arbitrage-free in the sense of NAFLVR.

Remark 3.1. (1) We want to point out that the present setting can be extended to consider the case where the numéraire X^0 is a general strictly positive semimartingale X^0 (not necessarily absolutely continuous), along the lines of [30]. In this case, one can obtain generalized versions of Theorems 3.2 and 3.11, at the expense of a more complex formulation.

(2) Under the additional assumption of locally bounded bond prices and right-continuity in maturity, [30, Theorem 5.2] shows that the existence of an equivalent local martingale measure is equivalent to the *no asymptotic free lunch (NAFL)* condition. However, in the general setting introduced in Section 2, such local boundedness and right-continuity properties are not necessarily satisfied.

3.2. The main result. In order to formulate our main theorem, we need to introduce some further notation. We set, for all $0 \leq t \leq T \leq \mathbb{T}$,

$$\begin{aligned} \bar{a}(t, T) &= \int_t^T a(t, u) du, & \bar{b}(t, T) &= \int_t^T b(t, u) du, \\ \bar{\alpha}(t, T) &= \int_{(t, T]} \alpha(t, u) \mu_t(du), & \bar{\beta}(t, T) &= \int_{(t, T]} \beta(t, u) \mu_t(du). \end{aligned}$$

Note that, as long as Assumptions 2.3 and 2.5 are satisfied, all the above integrals are well-defined. For each $T \in [0, \mathbb{T}]$, we introduce the process $Y^{(T)} = (Y_t^{(T)})_{0 \leq t \leq T}$ defined by

$$(3.1) \quad Y_t^{(T)} := \int_0^t \int_0^T g(s, u) \mu(ds, du), \quad \text{for all } 0 \leq t \leq T.$$

It will come as a consequence of Lemma 4.1 that $Y^{(T)}$ is a.s. finite and well-defined as a finite variation process. We denote by $\mu^{Y^{(T)}}$ the random jump measure of $Y^{(T)}$, in the sense of [24, Proposition II.1.16], with compensator $\mu^{p, Y^{(T)}}$. In view of [24, Theorem II.1.8], there exist an increasing integrable predictable process $A^{(T)} = (A_t^{(T)})_{0 \leq t \leq T}$ and a kernel $K^{(T)}(\omega, t; dy)$ from $(\Omega \times [0, \mathbb{T}], \mathcal{P})$ onto $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$(3.2) \quad \mu^{p, Y^{(T)}}(\omega; dt, dy) = K^{(T)}(\omega, t; dy) dA_t^{(T)}(\omega).$$

Note that, due to part (i) of Assumption 2.3, it holds that $\Delta Y_T^{(T)} = \int_0^T g(T, u) \mu(\{T\} \times du) = 0$. Hence, we may assume without loss of generality that $K^{(T)}(\omega, T; dy) = 0$, for all $(\omega, T) \in \Omega \times [0, \mathbb{T}]$. Let μ^p be the compensator of μ , which exists by part (ii) of Assumption 2.3 together with [24, Theorem II.1.8]. As shown in Lemma 4.3 below, the compensating measure $\mu^{p, Y^{(T)}}$ is linked to μ^p via the following relation, for all $0 \leq t \leq T \leq \mathbb{T}$:

$$(3.3) \quad \Delta A_t^{(T)} \int_{\mathbb{R}} y K^{(T)}(t; dy) = \int_{\mathbb{R}} y \mu^{p, Y^{(T)}}(\{t\} \times dy) = \int_{(t, T]} g(t, u) \mu^p(\{t\} \times du).$$

In addition, for each $T \in [0, \mathbb{T}]$, we let $\mu^{(Y^{(T)}, H)}$ be the random jump measure associated to the two-dimensional semimartingale $(Y^{(T)}, H)$ and $\mu^{p, (Y^{(T)}, H)}$ the corresponding compensator. Similarly as above, there exist an increasing integrable predictable process $B^{(T)} = (B_t^{(T)})_{0 \leq t \leq T}$ and a kernel $L^{(T)}(\omega, t; dy, dz)$ from $(\Omega \times [0, \mathbb{T}], \mathcal{P})$ onto $(\mathbb{R} \times \{0, 1\}, \mathcal{B}(\mathbb{R} \times \{0, 1\}))$ such that

$$(3.4) \quad \mu^{p, (Y^{(T)}, H)}(\omega; dt, dy, dz) = L^{(T)}(\omega, t; dy, dz) dB_t^{(T)}(\omega).$$

Finally, we define functions $W^{(1)} : \Omega \times [0, \mathbb{T}] \times \mathbb{R} \rightarrow \mathbb{R}$ and $W^{(2)} : \Omega \times [0, \mathbb{T}] \times \mathbb{R} \times \{0, 1\} \rightarrow \mathbb{R}$ as

$$(3.5) \quad \begin{aligned} W^{(1)}(\omega; s, y) &:= e^{g(\omega; s, s) \Delta \bar{\mu}_s(\omega)} (e^{-y} - 1), \\ W^{(2)}(\omega; s, y, z) &:= W^{(1)}(\omega; s, y) z. \end{aligned}$$

Note that, since the processes $(g(t, t))_{0 \leq t \leq \mathbb{T}}$ and $(\bar{\mu}_t)_{0 \leq t \leq \mathbb{T}}$ are predictable, the functions $W^{(1)}$ and $W^{(2)}$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ -measurable and $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \times \{0, 1\})$ -measurable, respectively.

Following the notation of [24], we denote by “ \star ” integration with respect to a random measure. Moreover, for an arbitrary process $V = (V_t)_{0 \leq t \leq \mathbb{T}}$ of finite variation, we denote by V^c its continuous part, which can be further decomposed as $V^c = \int_0^\cdot V_s^{ac} ds + V^{\text{sing}}$, similarly as in Lemma 2.1.

We are now in a position to state the following theorem, which gives necessary and sufficient conditions rendering the reference measure \mathbb{Q} a local martingale measure for the family $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, \mathbb{T}]\}$ with respect to the numéraire X^0 . As mentioned above, this represents the cornerstone for ensuring absence of arbitrage in the sense of NAFLVR. The proof of the theorem will be given in Section 4.2.

Theorem 3.2. *Suppose that Assumptions 2.3 and 2.5 hold. Let $W^{(1)}$ and $W^{(2)}$ be defined as in (3.5) and $g^{(T)}(\omega; s, u) := \mathbb{1}_{\{u \leq T\}}g(\omega; s, u)$, for each $T \in [0, \mathbb{T}]$. Then, the probability measure \mathbb{Q} is a local martingale measure for $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, \mathbb{T}]\}$ with respect to X^0 if and only if the following conditions hold a.s.:*

- (i) $f(t, t) + g(t, t)m_t = r_t + h_t$, for Lebesgue-a.e. $t \in [0, \mathbb{T}]$;
- (ii) $\Delta H_t^p = 1 - e^{-g(t, t)\Delta \bar{\mu}_t}$, for all $t \in [0, \mathbb{T}]$;
- (iii) $\Delta(W^{(1)} \star \mu^{p, Y^{(T)}})_t = \Delta(W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t$, for all $0 \leq t \leq T \leq \mathbb{T}$.
- (iv) for all $T \in [0, \mathbb{T}]$ and for Lebesgue-a.e. $t \in [0, T]$, it holds that

$$\begin{aligned} & -\bar{a}(t, T) - \bar{\alpha}(t, T) + \frac{1}{2}\|\bar{b}(t, T) + \bar{\beta}(t, T)\|^2 - (g^{(T)} \star \mu)_t^{\text{ac}} \\ & + (W^{(1)} \star \mu^{p, Y^{(T)}})_t^{\text{ac}} - (W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t^{\text{ac}} = 0; \end{aligned}$$

- (v) for all $0 \leq t \leq T \leq \mathbb{T}$, it holds that

$$\int_0^t g(s, s) d\nu_s - (g^{(T)} \star \mu)_t^{\text{sing}} + (W^{(1)} \star \mu^{p, Y^{(T)}})_t^{\text{sing}} - (W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t^{\text{sing}} = \lambda_t.$$

Condition (i) in Theorem 3.2 requires the instantaneous interest $f(t, t) + g(t, t)m_t$ accumulated by the credit risky bond be equal the risk-free rate of interest r_t plus a default risk compensation term given by h_t , which corresponds to the density of the absolutely continuous part of the default compensator H^p (see Lemma 2.1).

Condition (ii) is a precise matching condition between the jumps of the default compensator H^p and the jumps of the process $\bar{\mu}$ introduced in Lemma 2.4. In particular, letting the predictable times $\{U_i\}_{i \in \mathbb{N}}$ represent the jump times of H^p , condition (ii) implies that

$$(3.6) \quad \{(\omega, t) \in \Omega \times [0, \mathbb{T}] : g(\omega; t, t)\Delta \bar{\mu}_t(\omega) \neq 0\} = \{(\omega, t) \in \Omega \times [0, \mathbb{T}] : \Delta H_t^p(\omega) \neq 0\} = \bigcup_{i \in \mathbb{N}} \llbracket U_i \rrbracket.$$

Since the predictable stopping times $\{U_i\}_{i \in \mathbb{N}}$ correspond to possible default dates (i.e., it holds that $\mathbb{Q}(\tau = U_i \leq \mathbb{T}) > 0$, for all $i \in \mathbb{N}$) and the jumps of $\bar{\mu}$ correspond to possible “risky dates”, relation (3.6) means that “false alarms” (i.e., the possibility that a date for which there is no possibility of default is announced as a risky date) cannot happen. Moreover, observe that condition (ii) implies that, in order to exclude arbitrage, the credit risky term structure must exhibit discontinuities in maturity at the jump times $\{U_i\}_{i \in \mathbb{N}}$ of the default compensator. In other words, credit risky bond prices must be discontinuous in correspondence to the known risky dates (recall that $\Delta \bar{\mu}_t = \mu([0, \mathbb{T}] \times \{t\})$, for all $t \in [0, \mathbb{T}]$).

Condition (iii) essentially requires that the overall effect of new information about possible future risky periods arriving at predictable times and not coinciding with the default event vanishes. This can be seen by rewriting condition (iii) in the equivalent form

$$\begin{aligned} 0 &= \Delta(W^{(1)} \star \mu^{p, Y^{(T)}})_t - \Delta(W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t \\ &= e^{g(t, t)\Delta \bar{\mu}_t} \Delta B_t^{(T)} \int_{\mathbb{R} \times \{0, 1\}} (e^{-y} - 1)(1 - z)L^{(T)}(t; dy, dz) \\ &= \mathbb{E} \left[e^{g(t, t)\Delta \bar{\mu}_t} (e^{-\Delta Y_t^{(T)}} - 1)(1 - \Delta H_t) | \mathcal{F}_{t-} \right] \\ \Leftrightarrow & \mathbb{E} \left[\left(e^{-\int_t^T g(t, u)\mu(\{t\} \times du)} - 1 \right) (1 - \Delta H_t) | \mathcal{F}_{t-} \right] = 0, \end{aligned}$$

where we have used representation (3.4) together with [24, §II.1.11] and the predictability of the processes $(g(t, t))_{0 \leq t \leq T}$ and $\bar{\mu}$. Note that this condition is always satisfied if the process $Y^{(T)}$ is *quasi-left-continuous*, for every $T \in [0, T]$ (see [24, Corollary II.1.19]).

Condition (iv) represents the extension to a general defaultable setting of the classic HJM drift condition. In particular, the terms $W^{(1)} \star \mu^{p, Y^{(T)}}$ and $g^{(T)} \star \mu$ represent a compensation for the information received at time t concerning the likelihood of default in the future time period $(t, T]$, while the term $W^{(2)} \star \mu^{p, (Y^{(T)}, H)}$ accounts for the possibility of news arriving simultaneously to the default event.

Finally, condition (v) relates the continuous singular part λ of the default compensator H^p to the continuous singular parts of the processes appearing in the semimartingale decomposition of the term $\int_{(t, T]} g(t, u) \mu_t(du)$ introduced in (2.2). Note also that, making use of representations (3.2)-(3.4), the term $(W^{(1)} \star \mu^{p, Y^{(T)}})_t^{\text{sing}} - (W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t^{\text{sing}}$ appearing in condition (v) can be equivalently rewritten as

$$(W^{(1)} \star \mu^{p, Y^{(T)}})_t^{\text{sing}} - (W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t^{\text{sing}} = \int_0^t \int_{\mathbb{R} \times \{0, 1\}} (e^{-y} - 1)(1 - z) L^{(T)}(s; dy, dz) dB_s^{(T), \text{sing}}.$$

Remark 3.3 (On the impossibility of predictable default). Condition (ii) of Theorem 3.2 implies that $\Delta H_t^p < 1$ a.s., for all $t \in [0, T]$, since the term $g(t, t) \Delta \bar{\mu}_t$ is a.s. finite. In particular, this implies that the default time τ cannot be a *predictable time* (in the sense of [24, Definition I.2.7]). Intuitively, it is clear why a predictable default time is incompatible with an arbitrage-free term structure of the form (2.2), if $\int_t^T f(t, u) du + \int_{(t, T]} g(t, u) \mu_t(du) < +\infty$ a.s., for all $0 \leq t \leq T \leq T$. Indeed, if τ was a predictable time with $\mathbb{Q}(\tau \leq T) > 0$, for some $T \in [0, T]$, then the simple strategy $-1_{[\tau, T]}$ would realize an arbitrage opportunity, since $P(\tau, T) = 0$ and $P(\tau-, T) > 0$ hold on $\{\tau \leq T\}$. Note, however, that this does only exclude the case where the default time τ is a.s. equal to a predictable time, but does not exclude the case where τ can occur with strictly positive (but not unit) probability at predictable times.

3.3. Special cases and examples. Due to the generality of the setting, the conditions given in Theorem 3.2 are rather complex. In this section, we specialize the result of Theorem 3.2 to several situations and special cases of practical interest. Further examples related to the existing literature may be found in Section 3.5.

We start with the following simple lemma, which shows that conditions (iii)-(iv)-(v) of Theorem 3.2 can be simplified under a rather mild additional assumption on the news arrival process, encoded by the random measure μ .

Lemma 3.4. *Suppose that Assumptions 2.3 and 2.5 hold and assume furthermore that*

$$(3.7) \quad \mu(\{t\} \times [0, T]) \Delta H_t = 0 \text{ a.s. for all } t \in [0, T].$$

Then the term $W^{(2)} \star \mu^{p, (Y^{(T)}, H)}$ appearing in conditions (iii)-(iv)-(v) of Theorem 3.2 is null, up to an evanescent set, for every $T \in [0, T]$.

In particular, the above lemma shows that the term $W^{(2)} \star \mu^{p, (Y^{(T)}, H)}$ only plays the role of a compensation for the risk of news arriving simultaneously to the default event. Condition (3.7) can equivalently be phrased as a “no default by news” condition. In view of practical applications, this certainly represents a plausible assumption.

We now consider the case where condition (3.7) holds and the random measure $\mu(ds, du)$ is *integer-valued*, in the sense of [24, Definition II.1.13]. This additional assumption corresponds to

the situation where, at each date t , the new information arriving at that date only concerns a single time point (i.e., a possible “risky date”) in the future time period $(t, T]$. In view of practical applications, this case is still quite general and, as shown in the following corollary, allows for a substantial simplification of Theorem 3.2.

Corollary 3.5. *Suppose that Assumptions 2.3 and 2.5 hold and suppose furthermore that condition (3.7) holds and that the random measure $\mu(ds, du)$ is integer-valued. Let define the $\mathcal{P} \otimes \mathcal{B}([0, \mathbb{T}])$ -measurable function $\widehat{W}^{(T)}(\omega, s, u) := \mathbb{1}_{\{u \leq T\}} e^{g(\omega; s, s) \Delta \bar{\mu}_s(\omega)} (e^{-g(\omega; s, u)} - 1)$, for each $T \in [0, \mathbb{T}]$. Then \mathbb{Q} is a local martingale measure for $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, \mathbb{T}]\}$ with respect to X^0 if and only if the following conditions hold a.s.:*

- (i) $f(t, t) = r_t + h_t$, for Lebesgue-a.e. $t \in [0, \mathbb{T}]$;
- (ii) $\Delta H_t^p = 1 - e^{-g(t, t) \Delta \bar{\mu}_t}$, for all $t \in [0, \mathbb{T}]$;
- (iii) $\Delta(\widehat{W}^{(T)} \star \mu^p)_t = 0$, for all $0 \leq t \leq T \leq \mathbb{T}$;
- (iv) for all $T \in [0, \mathbb{T}]$ and Lebesgue-a.e. $t \in [0, T]$, it holds that

$$-\bar{a}(t, T) - \bar{\alpha}(t, T) + \frac{1}{2} \|\bar{b}(t, T) + \bar{\beta}(t, T)\|^2 + (\widehat{W}^{(T)} \star \mu^p)_t^{\text{ac}} = 0;$$

- (v) $(\widehat{W}^{(T)} \star \mu^p)_t^{\text{sing}} = \lambda_t$, for all $0 \leq t \leq T \leq \mathbb{T}$.

In particular, note that the additional assumptions that condition (3.7) holds and that μ is integer-valued imply that the default compensator can have a singular part if and only if the compensating measure μ^p admits a singular part (condition (v) of the above corollary). Furthermore, condition (i) simply requires the short end of the riskless forward rate $f(t, t)$ to be equal to the risk-free rate r_t plus the instantaneous compensation h_t for the risk of default. Observe also that the necessary and sufficient conditions appearing in the above corollary are directly formulated in terms of the compensating measure μ^p , without introducing the auxiliary jump measures $\mu^{p, Y^{(T)}}$ and $\mu^{p, (Y^{(T)}, H)}$. Moreover, if the compensator μ^p has the form $\mu^p(ds, du) = \xi_s(du)ds$, for some positive σ -finite measure $\xi_s(du)$ (this is the case if μ is a *homogeneous Poisson random measure*, see [24, Definition II.1.20]), then conditions (iii) and (v) of Corollary 3.5 are automatically satisfied if $\lambda = 0$ (compare also with Corollary 3.6 below).

As shown in the two following corollaries, Theorem 3.2 allows to recover the two special cases originally considered in [18]. The first corollary below considers a tractable setting where there is a finite set $\{U_1, \dots, U_N\}$ of random “risky dates”, at which default can happen with strictly positive probability and each of which is publicly announced at some previous time.

Corollary 3.6. *Suppose that Assumptions 2.3 and 2.5 hold and suppose furthermore that*

- (a) *the default compensator H^p satisfies $\lambda = 0$ and $\{\Delta H^p \neq 0\} = \bigcup_{i=1}^N \llbracket U_i \rrbracket$, for some $N \in \mathbb{N}$, and $\Delta H_{U_i}^p$ is \mathcal{F}_{S_i} -measurable, where $\{S_i\}_{i=1, \dots, N}$ is a sequence of strictly increasing stopping times such that $S_i < U_i$ a.s., for all $i = 1, \dots, N$;*
- (b) $\mu(ds, du) = \sum_{i=1}^N \delta_{\{S_i, U_i\}}(ds, du)$;
- (c) *the compensator μ^p has the form $\mu^p(ds, du) = \xi_s(du)ds$;*
- (d) $\mathbb{Q}(\tau = S_i) = 0$, for all $i = 1, \dots, N$.

Then \mathbb{Q} is a local martingale measure for $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, \mathbb{T}]\}$ with respect to X^0 if and only if the following conditions hold a.s.:

- (i) $f(t, t) = r_t + h_t$, for Lebesgue-a.e. $t \in [0, \mathbb{T}]$;
- (ii) $\Delta H_{U_i}^p = 1 - e^{-g(U_i, U_i)}$, for all $i = 1, \dots, N$;

(iii) for all $T \in [0, \mathbb{T}]$ and Lebesgue-a.e. $t \in [0, T]$, it holds that

$$-\bar{a}(t, T) - \bar{\alpha}(t, T) + \frac{1}{2} \|\bar{b}(t, T) + \bar{\beta}(t, T)\|^2 + \int_t^T (e^{-g(t, u)} - 1) \xi_t(du) = 0.$$

In particular, in comparison with the classical HJM drift condition, condition (iii) of Corollary 3.6 includes the additional term $\int_t^T (e^{-g(t, u)} - 1) \xi_t(du)$, which represents a compensation for the movements in the term structure due to the arrival of news concerning possible future risky dates.

The following example illustrates the modelling of bad news which may lead to discontinuities in the term structure. As pointed out in [18, 28], the failure of €1.5 billion of Greece on a scheduled debt repayment to the International Monetary fund as well as Argentina's missed coupon payment on \$29 billion debt are prominent examples of such cases.⁴

Example 3.7 (Sovereign credit with surprising bad news). Consider a credit from a country in the best rating class. Under normal circumstances, this could be interpreted as no default risk in the considered time horizon (i.e., $\tau = +\infty$). However, it might be the case that the country is hit by an unexpected event, which could be a catastrophe, a market crash or other unthought risks. Assume that news about this risk arrive at a random time U . The next expected payment of the credit is due at some random time $E > U$ and we denote the probability that the payment will be missed by $p \in [0, 1]$. Hence,

$$\tau = \begin{cases} E & \text{with probability } p; \\ +\infty & \text{with probability } 1 - p. \end{cases}$$

Let the filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq \mathbb{T}}$ be generated by the process $(\mathbb{1}_{\{U \leq t\}}(1 + E))_{0 \leq t \leq \mathbb{T}}$, properly augmented. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq \mathbb{T}}$ is given by the progressive enlargement of \mathbb{G} with τ , i.e.,

$$F_t = \bigcap_{s > t} (\mathcal{G}_s \vee \sigma(\tau \wedge s)), \quad \text{for all } 0 \leq t \leq \mathbb{T}.$$

Then, on $\{t < U\}$, no additional information is available and, hence, $\tau = +\infty$ with probability $(1 - p)$. Therefore, for all $A \in \mathcal{B}([0, \mathbb{T}])$,

$$\mathbb{1}_{\{t < U\}} \mathbb{Q}(\tau \in A | \mathcal{G}_t) = \mathbb{1}_{\{t < U\}} \left(p \mathbb{Q}(E \in A | \mathcal{G}_t) + (1 - p) \delta_\infty(A) \right) = \mathbb{1}_{\{t < U\}} \left(p \mathbb{Q}(E \in A) + (1 - p) \delta_\infty(A) \right).$$

Otherwise, on $\{t \geq U\}$, the risky date E is \mathcal{G}_t -measurable, so that

$$\mathbb{1}_{\{t \geq U\}} \mathbb{Q}(\tau \in A | \mathcal{G}_t) = \mathbb{1}_{\{t \geq U\}} \left(p \delta_E(A) + (1 - p) \delta_\infty(A) \right).$$

This example can be included in our framework by letting

$$\mu(ds, du) = \mathbb{1}_{\{U \leq s\}} \delta_E(du).$$

Assume, for simplicity, that the random variable E has a density. Then, credit risky bond prices following (2.2) turn out to be of classical HJM form, $\mathbb{1}_{\{\tau > t\}} e^{-\int_t^T f(t, u) du}$ for $t < U$, and

$$P(t, T) = \mathbb{1}_{\{\tau > t\}} \exp \left(- \int_t^T f(t, u) du - g(t, E) \mathbb{1}_{\{t < E \leq T\}} \right),$$

for $t \in [U, T]$. On the other hand, if the random variable E is discrete, one may consider the framework studied in the following result, Corollary 3.8. \diamond

⁴See e.g. the announcements in [23] and [35], as well as [11].

In the seminal model proposed by R. Merton in [32], debt of size K has to be repaid at some (deterministic) date $u_1 > 0$. Extensions to more sophisticated capital structures have been proposed, amongst others, in [19, 20]. In these cases, the credit structure may be incorporated by denoting the dates where obligatory payments are due by $0 < u_1 < \dots < u_N$ (such information is often publicly available⁵). Clearly, it is natural to expect discontinuities in the term structure at the dates $\{u_1, \dots, u_N\}$ and it does not pose any additional difficulty to consider infinitely many risky dates. The following corollary deals with this simple setting, to which we refer as *generalized Merton model*.

Corollary 3.8. *Suppose that Assumption 2.5 holds and suppose furthermore that*

- (a) *the default compensator H^P satisfies $\{\Delta H^P \neq 0\} = \bigcup_{i=1}^{\infty} \llbracket u_i \rrbracket$, where $\{u_i\}_{i \in \mathbb{N}}$ is a sequence of deterministic times, and the singular part λ is null;*
- (b) $\mu(ds, du) = \sum_{i=1}^{\infty} \delta_{(0, u_i)}(ds, du)$.

Then \mathbb{Q} is a local martingale measure for $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, \mathbb{T}]\}$ with respect to X^0 if and only if the following conditions hold a.s.:

- (i) $f(t, t) = r_t + h_t$, for Lebesgue a.-e. $t \in [0, \mathbb{T}]$;
- (ii) $\Delta H_{u_i}^P = 1 - e^{-g(u_i, u_i)}$, for all $i \in \mathbb{N}$;
- (iii) $\bar{a}(t, T) + \bar{\alpha}(t, T) = \frac{1}{2} \|\bar{b}(t, T) + \bar{\beta}(t, T)\|^2$, for all $T \in [0, \mathbb{T}]$ and Lebesgue-a.e. $t \in [0, T]$.

In particular, comparing condition (iii) of Corollary 3.6 with condition (iii) of Corollary 3.8, we see that there is no compensation for the arrival of news concerning possible future risky dates. This is simply due to the fact that, under the assumptions of Corollary 3.8, all risky dates are already publicly known at the initial date $t = 0$.

3.4. General recovery schemes. We have so far considered the case where the credit risky bond becomes worthless as soon as the default event occurs. In this section, taking up ideas from [2, 3, 8], we generalize the above framework to include general recovery schemes, where the credit risky bond is supposed to lose part of its value in correspondence of a sequence of default events. Before presenting the general theory, let us consider the following example.

Example 3.9 (Recovery of market value). Consider an \mathbb{F} -adapted marked point process $(\tau_n, e_n)_{n \in \mathbb{N}}$, meaning that $\{\tau_n\}_{n \in \mathbb{N}}$ are \mathbb{F} -stopping times and each random variable e_n is \mathcal{F}_{τ_n} -measurable. Each stopping time τ_n denotes a *default time* where the credit risky bond loses a fraction e_n of its market value. We assume that the fractional losses e_n take values in $[0, 1]$ (with the special case of zero recovery corresponding to $e_n = 1$). Note that, in line with [38], the loss at default e_n is possibly unpredictable, but known at the corresponding default time τ_n . Under this assumption (*fractional recovery of market value*), the term structure of credit risky bond prices can be assumed to be of the form

$$P(t, T) = \prod_{\tau_n \leq t} (1 - e_n) \cdot \exp \left(- \int_t^T f(t, u) du - \int_{(t, T]} g(t, u) \mu_t(du) \right), \quad \text{for all } 0 \leq t \leq T \leq \mathbb{T}.$$

Let us define the *recovery process* $\xi = (\xi_t)_{0 \leq t \leq \mathbb{T}}$ by

$$\xi_t = \prod_{n=1}^{\infty} \mathbb{1}_{\{\tau_n \leq t\}} (1 - e_n), \quad \text{for all } 0 \leq t \leq \mathbb{T}.$$

⁵See, for example <http://graphics.wsj.com/greece-debt-timeline/> for the debt structure of Greece collected by the Wall Street Journal.

Note that the recovery process ξ is adapted, starts at $\xi_0 = 1$ and is decreasing. In this example, strongly inspired by de-facto behavior of bond prices, the recovery process is piecewise constant. In the following, however, we shall allow for a more general behavior. \diamond

Inspired by the above example, let us now consider a general recovery process $\xi = (\xi_t)_{0 \leq t \leq \mathbb{T}}$ satisfying the following assumption.

Assumption 3.10. The recovery process $\xi = (\xi_t)_{0 \leq t \leq \mathbb{T}}$ is an \mathbb{F} -adapted càdlàg decreasing non-negative process with $\xi_0 = 1$.

Assumption 3.10 is clearly satisfied by the vast majority of recovery schemes typically considered in practice. In view of [21, Theorem 9.41], there exists a càdlàg decreasing process $R = (R_t)_{0 \leq t \leq \mathbb{T}}$ satisfying $-1 \leq \Delta R \leq 0$ such that $\xi = \mathcal{E}(R)$. We denote by μ^R the random jump measure of R and by $\mu^{p,R}$ its compensator. Since R admits limits from the left and has bounded jumps, it is locally bounded and, hence, special. [24, Corollary II.2.38] then implies that the process R admits the general representation

$$R_t = (x \star (\mu^R - \mu^{p,R}))_t - C_t, \quad \text{for all } 0 \leq t \leq \mathbb{T},$$

where $(C_t)_{0 \leq t \leq \mathbb{T}}$ is an increasing predictable process such that $\Delta C_t = -\int_{[-1,0]} x \mu^{p,R}(\{t\} \times dx)$, for all $t \in [0, \mathbb{T}]$.

Introducing the general recovery process $\xi = \mathcal{E}(R)$, we extend the term structure (2.2) as follows:

$$(3.8) \quad P(t, T) = \mathcal{E}(R)_t \exp \left(- \int_t^T f(t, u) du - \int_{(t, T]} g(t, u) \mu_t(du) \right), \quad \text{for all } 0 \leq t \leq T \leq \mathbb{T}.$$

The main goal of the present section consists in obtaining necessary and sufficient conditions for \mathbb{Q} to be a local martingale measure for the family $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, \mathbb{T}]\}$ with respect to the numéraire $X^0 = \exp(\int_0^\cdot r_t dt)$, thus extending Theorem 3.2 to general recovery schemes.

Letting the process $Y^{(T)}$ be defined as in (3.1), for every $T \in [0, \mathbb{T}]$, we denote by $\mu^{(Y^{(T)}, R)}$ the random jump measure associated to the two-dimensional semimartingale $(Y^{(T)}, R)$, with corresponding compensator $\mu^{p, (Y^{(T)}, R)}$. Additionally to the function $W^{(1)}$ introduced in (3.5), let us also introduce the function $W^{(3)} : \Omega \times [0, \mathbb{T}] \times \mathbb{R} \times [-1, 0] \rightarrow \mathbb{R}$ by

$$(3.9) \quad W^{(3)}(\omega; s, y, x) := W^{(1)}(\omega; s, y)x.$$

Note that the function $W^{(3)}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \times [-1, 0])$ -measurable. We are now in a position to state the following theorem. Similarly as above, we use the decomposition $C = \int_0^\cdot C_s^{\text{ac}} ds + C^{\text{sing}} + \sum_{0 < s \leq \cdot} \Delta C_s$, with C^{ac} and C^{sing} denoting respectively the density of the absolutely continuous part and the singular part of C^c (compare also with Lemma 2.1).

Theorem 3.11. Suppose that Assumptions 2.3, 2.5 and 3.10 hold. Let $W^{(1)}$ and $W^{(3)}$ be defined as in (3.5) and (3.9), respectively, and $g^{(T)}(\omega; s, u) := \mathbb{1}_{\{u \leq T\}} g(\omega; s, u)$, for all $T \in [0, \mathbb{T}]$. Then, the probability measure \mathbb{Q} is a local martingale measure for $\{(P(t, T))_{0 \leq t \leq T}; T \in [0, \mathbb{T}]\}$ with respect to X^0 if and only if the following conditions hold a.s.:

- (i) $f(t, t) + g(t, t)m_t = r_t + C_t^{\text{ac}}$ for Lebesgue-a.e. $t \in [0, \mathbb{T}]$;
- (ii) $\Delta C_t = 1 - e^{-g(t, t)\Delta \bar{\mu}_t}$, for all $t \in [0, \mathbb{T}]$;
- (iii) $\Delta(W^{(1)} \star \mu^{p, Y^{(T)}})_t = -\Delta(W^{(3)} \star \mu^{p, (Y^{(T)}, R)})_t$, for all $0 \leq t \leq T \leq \mathbb{T}$;
- (iv) for all $T \in [0, \mathbb{T}]$ and for Lebesgue-a.e. $t \in [0, T]$, it holds that

$$-\bar{a}(t, T) - \bar{\alpha}(t, T) + \frac{1}{2} \|\bar{b}(t, T) + \bar{\beta}(t, T)\|^2 - (g^{(T)} \star \mu)_t^{\text{ac}}$$

$$+ (W^{(1)} \star \mu^{p, Y^{(T)}})_t^{\text{ac}} + (W^{(3)} \star \mu^{p, (Y^{(T)}, R)})_t^{\text{ac}} = 0;$$

(v) for all $0 \leq t \leq T \leq \mathbb{T}$, it holds that

$$\int_0^t g(s, s) d\nu_s - (g^{(T)} \star \mu)_t^{\text{sing}} + (W^{(1)} \star \mu^{p, Y^{(T)}})_t^{\text{sing}} + (W^{(3)} \star \mu^{p, (Y^{(T)}, R)})_t^{\text{sing}} = C_t^{\text{sing}}.$$

The interpretation of the five conditions stated in the above theorem is analogous to the case of Theorem 3.2.

3.5. Related literature. As mentioned in the introduction, the two classical approaches to credit risk are the *structural* approach, starting with Merton [32] and its extensions [19, 20], and the *reduced-form* approach, introduced in early works of Jarrow, Lando and Turnbull [27, 31] and in [1]. It was a long time that these approaches co-existed in the literature but no model was bridging these two classes (see however [26] for an information-based perspective connecting structural and reduced-form models). The remarkable paper [2] considered a first-passage-time model over a random boundary for the default time and pointed towards an extension of the reduced-form approach beyond intensity-based models. The framework may be seen as a structural approach where the debt level is random and we give a short account.

3.5.1. The relation to Bélanger, Shreve and Wong (2004). In [2], the authors consider a background filtration \mathbb{G} , given by the augmented filtration generated by a Brownian motion W . Additionally, there is a càdlàg non-decreasing \mathbb{G} -predictable process $(\Lambda_t)_{0 \leq t \leq \mathbb{T}}$ and the default time τ is defined as

$$\tau := \inf\{t \in [0, \mathbb{T}] : \Lambda_t \geq \Theta\},$$

where Θ is a strictly positive random variable independent of \mathbb{G} . The filtration \mathbb{F} is then defined as the progressive enlargement of \mathbb{G} with respect to τ . Depending on the choice of the process $(\Lambda_t)_{0 \leq t \leq \mathbb{T}}$, it is shown that the default compensator H^p may contain jumps as well as a singular continuous part, thus exploiting the generality of our decomposition (2.1) and going beyond classical intensity-based models. However, the HJM approach to the modelling of defaultable term structures is only considered in [2, Section 5] in an intensity-based setting (i.e., assuming that the default compensator H^p is absolutely continuous).

3.5.2. The relation to Jiao and Li (2015). More recently, extensions of the intensity-based approach have been pursued via methods of enlargements of filtration, see [14, 15]. This approach has been extended in [28] to the case where the default compensator exhibits discontinuities. Starting from a background filtration \mathbb{G} , [28] consider a finite family $\{\tau_1, \dots, \tau_n\}$ of \mathbb{G} -stopping times, which can be chosen strictly increasing without loss of generality. The filtration \mathbb{F} is then constructed as the progressive enlargement of \mathbb{G} with respect to τ . Letting $(\alpha_t)_{0 \leq t \leq \mathbb{T}}$ be a \mathbb{G} -optional process taking values in the space of measurable functions on \mathbb{R}_+ , [28] propose the following *generalized density hypothesis*:

$$\mathbb{E} \left[\mathbb{1}_{\{\tau < +\infty\}} h(\tau) \prod_{i=1}^n \mathbb{1}_{\{\tau \neq \tau_i\}} \middle| \mathcal{G}_t \right] = \int_{\mathbb{R}_+} h(u) \alpha_t(u) \eta(du) \quad \text{a.s. for all } 0 \leq t \leq \mathbb{T},$$

for any bounded measurable function $h(\cdot)$, where η is assumed to be a non-atomic σ -finite Borel measure on \mathbb{R}_+ . In [28, Section 3], the default compensator H^p is computed under this hypothesis. It is shown that H^p contains an integral with respect to the measure η , which may not necessarily be absolutely continuous and, in addition, H^p depends on the \mathbb{G} -compensators of the processes

$\mathbb{1}_{\llbracket \tau_i, +\infty \rrbracket}$ which are allowed to be fully general and may therefore exhibit a jumping behavior. Clearly, this specification can be covered by the general decomposition (2.1).

As an example, [28] consider the case $\tau := \tau_1 \wedge E$, where E is exponentially distributed and τ_1 is the first passage time of a Brownian motion below the level $a < 0$. In this case it, follows that

$$(3.10) \quad H_t^p = \int_0^t h_s ds + \mathbb{1}_{\{\tau_1 \leq t\}} \Delta H_{\tau_1}^p, \quad \text{for all } 0 \leq t \leq \tau.$$

Our results can be applied to this setting and permit to describe the general class of arbitrage-free term structure models compatible with this structure of the default compensator.

The approach of [28] has been recently extended in the context of sovereign default risk in [29]. Consider a sequence of increasing levels $0 < a_1 < a_2 < \dots < a_n$ and denote by $\{\tau_i\}_{i=1,\dots,n}$ the (increasing) first-passage-times of a Brownian motion of these levels. Let E' be an independent exponentially distributed random variable. The times τ_i represent critical political events where the sovereign seeks financial aid to avoid immediate default. If $\tau_i > E'$, this attempt was not successful and default occurs. Furthermore, [29] consider an additional doubly-stochastic random time with an intensity and let the default time τ be the minimum of such times. In summary, the authors show that $H^p = \int_0^\cdot h_s ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq \cdot\}} \Delta H_{\tau_i}^p$, for all $0 \leq t \leq \tau$. The authors also study the case where the Brownian motion is replaced by a diffusive Markov process and obtain explicit formulas for geometric Brownian motion and for the CEV process.

4. PROOFS

This section contains the proofs of all the results stated in Section 3. After giving the proof of the two technical lemmata contained in Section 2, we present the proof of Theorem 3.2, while the proofs of the results stated in Section 3.3 are given in Section 4.3. Finally, Section 4.4 presents the proof of Theorem 3.11.

4.1. Proofs of the results stated in Section 2.

Proof of Lemma 2.1. Since H^p is a predictable process of finite variation, it can be decomposed as $H^p = (H^p)^c + \sum_{0 < s \leq \cdot} \Delta H_s^p$, where $(H^p)^c$ is an increasing continuous process. Theorem 2.1 in [7] then yields the existence of an integrable predictable process $(h_t)_{0 \leq t \leq \mathbb{T}}$ such that

$$(H^p)^c_t = \int_0^t h_s ds + \int_0^t \mathbb{1}_N(s) d(H^p)_s^c, \quad \text{for all } 0 \leq t \leq \mathbb{T},$$

where N is a predictable subset of $\Omega \times [0, \mathbb{T}]$ such that the sections $N_\omega = \{t \in [0, \mathbb{T}] : (\omega, t) \in N\}$ have Lebesgue measure zero, for a.a. $\omega \in \Omega$. The result follows by letting $\lambda := \int_0^\cdot \mathbb{1}_N(s) d(H^p)_s^c$. \square

Proof of Lemma 2.4. Note first that, due to parts (i)-(ii) of Assumption (2.3), it holds that, for every $t \in [0, \mathbb{T}]$,

$$\bar{\mu}_t = \mu([0, \mathbb{T}] \times [0, t]) = \mu([0, t] \times [0, t]) = \int_0^t \int_0^{\mathbb{T}} \mathbb{1}_{\{u \leq t\}} \mu(ds, du).$$

For any measurable bounded function $\vartheta : [0, \mathbb{T}] \times [0, \mathbb{T}] \rightarrow \mathbb{R}_+$, the process $(\Theta(t, v))_{0 \leq t \leq \mathbb{T}}$ defined by $\Theta(t, v) := \int_0^t \int_0^{\mathbb{T}} \vartheta(v, u) \mu(ds, du)$, is optional and increasing, for every $v \in [0, \mathbb{T}]$, since $\mu(ds, du)$ is a non-negative optional random measure. Being increasing, $(\Theta(t, v))_{0 \leq t \leq \mathbb{T}}$ admits limits from the left, so that the process $(\Theta(t-, v))_{0 \leq t \leq \mathbb{T}}$ is adapted and left-continuous, hence predictable, for every $v \in [0, \mathbb{T}]$. In turn, this implies that the processes $(\Theta(t, t))_{0 \leq t \leq \mathbb{T}}$ and $(\Theta(t-, t))_{0 \leq t \leq \mathbb{T}}$ are optional and predictable, respectively. Indeed, this is obvious for functions of the form $\vartheta(t, u) =$

$p(t)q(u)$, with $p, q : [0, \mathbb{T}] \rightarrow \mathbb{R}_+$ bounded and measurable (since any deterministic function is clearly predictable) and the general case then follows by a monotone class argument. Letting $\vartheta(t, u) = \mathbb{1}_{\{u \leq t\}}$, this shows that the process $(\bar{\mu}_t)_{0 \leq t \leq \mathbb{T}}$ is optional and increasing. Moreover, due to part (i) of Assumption (2.3), it holds that

$$\bar{\mu}_t = \mu([0, t] \times [0, t]) = \mu([0, t] \times [0, t]) = \Theta(t-, t),$$

thus showing the predictability of $\bar{\mu}$. An application of the dominated convergence theorem (for each $\omega \in \Omega$), together with part (ii) of Assumption 2.3, allows to show that μ is also right-continuous. The decomposition (2.5) follows by the same arguments used in the proof of Lemma 2.1. \square

4.2. Proof of Theorem 3.2. Since the proof of Theorem 3.2 requires several intermediate steps, let us first give an outline of the main ideas involved. The starting point consists in representing the pre-default price (i.e., on the event $\{H_t = 0\}$) of a credit risky bond as an exponential of a semimartingale admitting an explicit decomposition into a predictable finite variation part, a continuous local martingale part and an integral with respect to the random measure μ . As a second step, we conveniently transform the ordinary exponential into a stochastic exponential. The desired local martingale property of (discounted) credit risky bond prices will then be equivalent to the local martingale property of the process defining the stochastic exponential. By computing the canonical decomposition of the latter, the local martingale property then holds if and only if all predictable finite variation terms vanish. This will then lead to the conditions stated in Theorem 3.2.

We start by rewriting the defaultable bond price $P(t, T)$ in the following form:

$$(4.1) \quad P(t, T) = (1 - H_t)F(t, T)G(t, T),$$

where

$$F(t, T) := \exp\left(-\int_t^T f(t, u)du\right) \quad \text{and} \quad G(t, T) := \exp\left(-\int_{(t, T]} g(t, u)\mu_t(du)\right),$$

for all $0 \leq t \leq T \leq \mathbb{T}$. By Assumption 2.5 and following the original arguments of [22], the term $F(t, T)$ admits the representation

$$(4.2) \quad F(t, T) = \exp\left(\int_0^t f(s, s)ds - \int_0^t \bar{a}(s, T)ds - \int_0^t \bar{b}(s, T)dW_s\right),$$

see, e.g., [17, Lemma 6.1] (note that, in comparison to this work, we rely on a weaker assumption on the volatility process b for the application of the stochastic Fubini theorem by virtue of [34, Theorem IV.65] or [4, Proposition A.2]).

The next lemma, which extends [18, Lemma 2.3] to the present general setting, derives a representation analogous to (4.2) for the term $G(t, T)$.

Lemma 4.1. *Suppose that Assumptions 2.3 and 2.5 hold. Then, for each $T \in [0, \mathbb{T}]$, the process $(\log G(t, T))_{0 \leq t \leq T}$ is a semimartingale admitting the decomposition*

$$\log G(t, T) = \int_0^t g(s, s) d\bar{\mu}_s - \int_0^t \bar{\alpha}(s, T)ds - \int_0^t \bar{\beta}(s, T)dW_s - \int_0^t \int_{(s, T]} g(s, u)\mu(ds, du).$$

Proof. We first show that the stochastic integral $\int_0^\cdot \bar{\beta}(s, T) dW_s$ is well-defined, for every $T \in [0, \mathbb{T}]$. To this effect, observe first that part (iii) of Assumption 2.5 implies that $(\sqrt{\int_0^\mathbb{T} (\beta^i(s, u))^2 \mu_s(du)})_{0 \leq s \leq T}$ belongs to $L(W^i; \tilde{\mathbb{F}})$, for all $i = 1, \dots, n$. By part (ii) of Assumption 2.3, the \mathbb{F} -Brownian motion W is a continuous $\tilde{\mathbb{F}}$ -semimartingale and, due to Lévy's characterization theorem, its continuous $\tilde{\mathbb{F}}$ -martingale part is necessarily an $\tilde{\mathbb{F}}$ -Brownian motion. Hence, recalling that a process is integrable with respect to a continuous semimartingale if and only if it is integrable with respect to both components of the canonical decomposition of the latter (see, e.g., [24, Proposition III.6.22]), this implies that $(\sqrt{\int_0^\mathbb{T} (\beta^i(s, u))^2 \mu_s(du)})_{0 \leq s \leq \mathbb{T}}$ is integrable with respect to an $\tilde{\mathbb{F}}$ -Brownian motion, so that

$$\int_0^\mathbb{T} \int_0^\mathbb{T} \|\beta(s, u)\|^2 \mu_s(du) ds < +\infty \text{ a.s.}$$

Hölder's inequality, together with Assumptions 2.3 and 2.5, then implies that, for every $T \in [0, \mathbb{T}]$,

$$\begin{aligned} \int_0^T \|\bar{\beta}(s, T)\|^2 ds &= \int_0^T \left\| \int_{(s, T]} \beta(s, u) \mu_s(du) \right\|^2 ds \leq \int_0^T \left(\mu_s([0, T]) \int_0^T \|\beta(s, u)\|^2 \mu_s(du) \right) ds \\ &\leq \mu_T([0, T]) \int_0^T \int_0^T \|\beta(s, u)\|^2 \mu_s(du) ds < +\infty \text{ a.s.} \end{aligned}$$

thus proving the well-posedness of the stochastic integral $\int_0^\cdot \bar{\beta}(s, T) dW_s$. In turn, since the term $G(t, T)$ is a.s. finite for every $0 \leq t \leq T \leq \mathbb{T}$, as a consequence of part (ii) of Assumption 2.3 together with the continuity of the map $u \mapsto g(\omega; t, u)$, this implies that the term $\int_0^t \int_{(s, T]} g(s, u) \mu(ds, du) = Y_t^{(T)}$ is a.s. finite and, hence, well-defined as a finite variation process.

Observe then that, by the definition of $\mu_t(du)$,

$$(4.3) \quad -\log G(t, T) = \int_{(t, T]} g(t, u) \mu_t(du) = \int_0^t \int_{(t, T]} g(t, u) \mu(ds, du) = \int_0^t \int_0^T \mathbf{1}_{\{u > t\}} g(t, u) \mu(ds, du).$$

The product rule, together with equation (2.4) and the continuity of g , yields that

$$(4.4) \quad \begin{aligned} \mathbf{1}_{[0, u)}(t) g(t, u) &= g(0, u) + \int_0^t \mathbf{1}_{[0, u)}(v) dg(v, u) + \int_0^t g(v, u) d(\mathbf{1}_{[0, u)}(v)) \\ &= g(0, u) + \int_0^t \mathbf{1}_{[0, u)}(v) \alpha(v, u) dv + \int_0^t \mathbf{1}_{[0, u)}(v) \beta(v, u) dW_v - g(u, u) \mathbf{1}_{\{u \leq t\}}, \end{aligned}$$

where both integrals are well-defined by Assumption 2.5. Equations (4.3)-(4.4) imply that

$$(4.5) \quad \begin{aligned} \int_{(t, T]} g(t, u) \mu_t(du) &= \int_0^t \int_0^T g(0, u) \mu(ds, du) + \int_0^t \int_0^T \int_0^t \mathbf{1}_{[0, u)}(v) \alpha(v, u) dv \mu(ds, du) \\ &\quad + \int_0^t \int_0^T \int_0^t \mathbf{1}_{[0, u)}(v) \beta(v, u) dW_v \mu(ds, du) - \int_0^t \int_0^T g(u, u) \mathbf{1}_{\{u \leq t\}} \mu(ds, du) \\ &=: (1) + (2) + (3) + (4). \end{aligned}$$

Due to part (ii) of Assumption 2.5, we can apply for each $\omega \in \Omega$ the classical Fubini theorem to the term (2), so that

$$\begin{aligned} (2) &= \int_0^t \int_0^T \int_0^s \mathbf{1}_{[0, u)}(v) \alpha(v, u) dv \mu(ds, du) + \int_0^t \int_0^T \int_s^t \mathbf{1}_{[0, u)}(v) \alpha(v, u) dv \mu(ds, du) \\ &= \int_0^t \int_0^T \int_0^s \mathbf{1}_{[0, u)}(v) \alpha(v, u) dv \mu(ds, du) + \int_0^t \int_0^v \int_0^T \mathbf{1}_{[0, u)}(v) \alpha(v, u) \mu(ds, du) dv. \end{aligned}$$

Assumption 2.3 together with part (iii) of Assumption 2.5 allows to perform an analogous change of the order of integration with respect to dW_v and $\mu(ds, du)$ in the term (3). Indeed, due to part (ii) of Assumption 2.3, by localization we can assume that the random measure μ is integrable, i.e., $\mathbb{E}[\mu_T([0, T])] < +\infty$, for all $T \in [0, T]$. Then, letting $X = W^i$, for each $i = 1, \dots, n$, and taking $E = [0, T] \times [0, T]$, $\varrho(\omega; dx) = \mu(\omega; ds, du)$ and $H_t(\omega; s, u) = \mathbb{1}_{[0, t]}(s)\mathbb{1}_{(t, T]}(u)\beta(\omega; t, u)$, Assumptions 2.3 and 2.5 imply that the hypotheses of Proposition 5.1 are satisfied (for each component of the n -dimensional Brownian motion W), so that

$$(3) = \int_0^t \int_0^T \int_0^s \mathbb{1}_{[0, u)}(v)\beta(v, u) dW_v \mu(ds, du) + \int_0^t \int_0^v \int_0^T \mathbb{1}_{[0, u)}(v)\beta(v, u)\mu(ds, du) dW_v.$$

In view of Proposition 5.1, the stochastic integral appearing in the second term on the right-hand side of the last equation has to be understood a priori in the enlarged filtration $\tilde{\mathbb{F}}$. However, the integrand $\int_0^v \int_0^T \mathbb{1}_{[0, u)}(v)\beta(v, u)\mu(ds, du) = \int_{(v, T]} \beta(v, u)\mu_v(du)$ is adapted to the original filtration \mathbb{F} , since the random measure μ is \mathbb{F} -optional and β is $\mathcal{O} \otimes \mathcal{B}([0, T])$ -measurable. Hence, in view of [24, Proposition III.6.25], the stochastic integral is also well-defined in the original filtration \mathbb{F} and coincides with the same stochastic integral viewed in the enlarged filtration $\tilde{\mathbb{F}}$. Note also that

$$\begin{aligned} \int_0^t \int_0^v \int_0^T \mathbb{1}_{[0, u)}(v)\alpha(v, u)\mu(ds, du) dv &= \int_0^t \int_{(v, T]} \alpha(v, u)\mu_v(du) dv = \int_0^t \bar{\alpha}(v, T) dv, \\ \int_0^t \int_0^v \int_0^T \mathbb{1}_{[0, u)}(v)\beta(v, u)\mu(ds, du) dW_v &= \int_0^t \int_{(v, T]} \beta(v, u)\mu_v(du) dW_v = \int_0^t \bar{\beta}(v, T) dW_v, \end{aligned}$$

where both integrals are well-defined by Assumption 2.5. Moreover, due to equation (4.4), it holds that

$$\int_0^s \mathbb{1}_{[0, u)}(v)\alpha(v, u)dv + \int_0^s \mathbb{1}_{[0, u)}(v)\beta(v, u)dW_v = \mathbb{1}_{[0, u)}(s)g(s, u) - g(0, u) + g(u, u)\mathbb{1}_{\{u \leq s\}},$$

so that equation (4.5) can be rewritten as

$$\begin{aligned} \int_{(t, T]} g(t, u)\mu_t(du) &= \int_0^t \bar{\alpha}(v, T)dv + \int_0^t \bar{\beta}(v, T)dW_v \\ &\quad + \int_0^t \int_0^T \mathbb{1}_{[0, u)}(s)g(s, u)\mu(ds, du) - \int_0^t \int_0^T \mathbb{1}_{(s, t]}(u)g(u, u)\mu(ds, du). \end{aligned}$$

Finally, part (i) of Assumption 2.3 and the definition of the process $\bar{\mu}$ in Lemma 2.4 imply that

$$\begin{aligned} \int_0^t \int_0^T \mathbb{1}_{(s, t]}(u)g(u, u)\mu(ds, du) &= \int_0^t \int_{(s, t]} g(u, u)\mu(ds, du) = \int_0^t \int_0^t g(u, u)\mu(ds, du) \\ &= \int_0^T \int_0^t g(u, u)\mu(ds, du) = \int_0^t g(u, u) d\bar{\mu}_u. \end{aligned}$$

As already remarked at the end of Section 2.4, the process $\int_0^\cdot g(s, s) d\bar{\mu}_s$ is predictable and of finite variation. This implies the semimartingale property of the process $(\log G(t, T))_{0 \leq t \leq T}$, for every $T \in [0, T]$. \square

For each $T \in [0, T]$, let us then define the process $(X_t^{(T)})_{0 \leq t \leq T}$ by

$$\begin{aligned} (4.6) \quad X_t^{(T)} &:= \log(F(t, T)) + \log(G(t, T)) \\ &= \int_0^t f(s, s)ds - \int_0^t \bar{a}(s, T)ds - \int_0^t \bar{b}(s, T)dW_s \\ &\quad + \int_0^t g(s, s)d\bar{\mu}_s - \int_0^t \bar{\alpha}(s, T)ds - \int_0^t \bar{\beta}(s, T)dW_s - \int_0^t \int_{(s, T]} g(s, u)\mu(ds, du), \end{aligned}$$

so that $P(t, T) = (1 - H_t) \exp(X_t^{(T)})$. In the following lemma, we give an alternative representation of the defaultable bond price $P(t, T)$ as a stochastic exponential.

Lemma 4.2. *Suppose that Assumptions 2.3 and 2.5 hold. Then, for each $0 \leq t \leq T \leq \mathbb{T}$, the credit risky bond price $P(t, T)$ can be represented as*

$$(4.7) \quad P(t, T) = \mathcal{E} \left(\tilde{X}^{(T)} - H - [\tilde{X}^{(T)}, H] \right)_t,$$

where, for each $T \in [0, \mathbb{T}]$, the process $(\tilde{X}_t^{(T)})_{0 \leq t \leq T}$ is defined as

$$(4.8) \quad \begin{aligned} \tilde{X}_t^{(T)} := & X_t^{(T)} + \frac{1}{2} \int_0^t \|\bar{b}(s, T) + \bar{\beta}(s, T)\|^2 ds \\ & + \sum_{0 < s \leq t} \left(e^{-\int_s^T g(s, u) \mu(\{s\} \times du) + g(s, s) \Delta \bar{\mu}_s} - 1 + \int_s^T g(s, u) \mu(\{s\} \times du) - g(s, s) \Delta \bar{\mu}_s \right). \end{aligned}$$

Proof. Since the process $(H_t)_{0 \leq t \leq \mathbb{T}}$ is a single jump process with jump size equal to one, it follows that, by the definition of stochastic exponential,

$$1 - H_t = e^{H_0 - H_t} \prod_{0 < s \leq t} (1 - \Delta H_s) e^{\Delta H_s} = \mathcal{E}(-H)_t.$$

Moreover, [24, Theorem II.8.10] implies that $\exp(X_t^{(T)}) = \mathcal{E}(\tilde{X}^{(T)})_t$, for all $0 \leq t \leq T \leq \mathbb{T}$, where the process $\tilde{X}^{(T)}$ is defined as in (4.8). The representation (4.7) then follows by Yor's formula (see, e.g., [24, §II.8.19]). \square

Our next goal consists in developing a more tractable representation of the process defining the stochastic exponential in (4.7). To this effect, let us analyze in more detail the jumps of the semimartingale $\tilde{X}^{(T)} - H - [\tilde{X}^{(T)}, H]$:

$$(4.9) \quad \begin{aligned} \Delta(\tilde{X}^{(T)} - H - [\tilde{X}^{(T)}, H])_t &= \Delta \tilde{X}_t^{(T)} - \Delta H_t - \Delta H_t \Delta \tilde{X}_t^{(T)} \\ &= e^{-\int_t^T g(t, u) \mu(\{t\} \times du) + g(t, t) \Delta \bar{\mu}_t} - 1 - \Delta H_t \\ &\quad - \Delta H_t \left(e^{-\int_t^T g(t, u) \mu(\{t\} \times du) + g(t, t) \Delta \bar{\mu}_t} - 1 \right) \\ &= e^{g(t, t) \Delta \bar{\mu}_t} \left(e^{-\int_t^T g(t, u) \mu(\{t\} \times du)} - 1 \right) + \left(e^{g(t, t) \Delta \bar{\mu}_t} - 1 \right) (1 - \Delta H_t) \\ &\quad - \Delta H_t - e^{g(t, t) \Delta \bar{\mu}_t} \left(e^{-\int_t^T g(t, u) \mu(\{t\} \times du)} - 1 \right) \Delta H_t. \end{aligned}$$

Let us rewrite this last expression in a more compact way by using the notation introduced in Section 3.2. To this effect, for each $T \in [0, \mathbb{T}]$, we make use of the process $Y^{(T)} = \int_0^T \int_0^T g(s, u) \mu(ds, du)$ and the corresponding jump measure $\mu^{Y^{(T)}}$, so that

$$(4.10) \quad \sum_{0 < s \leq t} e^{g(s, s) \Delta \bar{\mu}_s} \left(e^{-\int_s^T g(s, u) \mu(\{s\} \times du)} - 1 \right) = (W^{(1)} \star \mu^{Y^{(T)}})_t,$$

with $W^{(1)}(\omega, s, y) = e^{g(\omega; s, s) \Delta \bar{\mu}_s(\omega)} (e^{-y} - 1)$, as introduced in (3.5). Note that the function $W^{(1)}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ -measurable and $W^{(1)} \star \mu^{Y^{(T)}}$ makes sense as an integral with respect to the random measure $\mu^{Y^{(T)}}$. Indeed, (4.10) is well-defined, since

$$(4.11) \quad \begin{aligned} & \sum_{0 < s \leq t} e^{g(s, s) \Delta \bar{\mu}_s} \left| e^{-\int_s^T g(s, u) \mu(\{s\} \times du)} - 1 \right| \\ & \leq \sum_{0 < s \leq t} \left| e^{-\int_s^T g(s, u) \mu(\{s\} \times du) + g(s, s) \Delta \bar{\mu}_s} - 1 \right| + \sum_{0 < s \leq t} \left| e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right| < +\infty \text{ a.s.} \end{aligned}$$

In fact, the first sum appearing in (4.11) is finite as a consequence of (4.8) together with the fact that the two processes $Y^{(T)}$ and $\int_0^T g(s, s) d\bar{\mu}_s$ are of finite variation, which in turn implies that $\sum_{0 < s \leq t} \int_s^T g(s, u) \mu(\{s\} \times du)$ and $\sum_{0 < s \leq t} g(s, s) \Delta \bar{\mu}_s$ are a.s. finite. Moreover, the process $\int_0^T g(s, s) d\bar{\mu}_s$ is predictable and of finite variation, hence also locally bounded (see, e.g., [24, Lemma I.3.10]) and exponentially special, so that the second term appearing in (4.11) is a.s. finite by [24, Proposition II.8.26]. In particular, this also ensures that the summations of the jump terms appearing in (4.9) are well-defined and a.s. finite. Similarly, using the jump measure $\mu^{(Y^{(T)}, H)}$ of the semimartingale $(Y^{(T)}, H)$, we obtain

$$\sum_{0 < s \leq t} e^{g(s, s) \Delta \bar{\mu}_s} \left(e^{-\int_s^T g(s, u) \mu(\{s\} \times du)} - 1 \right) \Delta H_s = (W^{(2)} \star \mu^{(Y^{(T)}, H)})_t,$$

with $W^{(2)}(\omega, s, y, z) = e^{g(\omega; s, s) \Delta \bar{\mu}_s(\omega)} (e^{-y} - 1)z$, as defined in (3.5) (note that this summation does not pose any integrability problem since H is a single jump process). We have thus obtained the representation:

$$\begin{aligned} \sum_{0 < s \leq t} \Delta(\tilde{X}^{(T)} - H - [\tilde{X}^{(T)}, H])_s &= \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) (1 - \Delta H_s) - H_t \\ &\quad + (W^{(1)} \star \mu^{Y^{(T)}})_t - (W^{(2)} \star \mu^{(Y^{(T)}, H)})_t. \end{aligned}$$

In turn, together with the definition of the process $\tilde{X}^{(T)}$ (see (4.8)) and decomposition (2.5), this implies that the semimartingale $\tilde{X}^{(T)} - H - [\tilde{X}^{(T)}, H]$ defining the stochastic exponential (4.7) admits the following decomposition:

$$\begin{aligned} \tilde{X}_t^{(T)} - H_t - [\tilde{X}^{(T)}, H]_t &= \int_0^t f(s, s) ds - \int_0^t \bar{a}(s, T) ds - \int_0^t \bar{\alpha}(s, T) ds \\ &\quad + \frac{1}{2} \int_0^t \|\bar{b}(s, T) + \bar{\beta}(s, T)\|^2 ds + \int_0^t g(s, s) m_s ds + \int_0^t g(s, s) d\nu_s \\ &\quad - \int_0^t \bar{b}(s, T) dW_s - \int_0^t \bar{\beta}(s, T) dW_s - (g^{(T)} \star \mu)_t^c \\ &\quad + \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) (1 - \Delta H_s) - H_t \\ &\quad + (W^{(1)} \star \mu^{Y^{(T)}})_t - (W^{(2)} \star \mu^{(Y^{(T)}, H)})_t, \end{aligned} \tag{4.12}$$

where $g^{(T)}(\omega; s, u) := \mathbf{1}_{\{u \leq T\}} g(\omega; s, u)$ and $(g^{(T)} \star \mu)^c$ denotes the continuous part of the finite variation process $g^{(T)} \star \mu = Y^{(T)}$.

We are now in a position to do the final step of the proof Theorem 3.2.

Proof of Theorem 3.2. Recall that, in view of Lemma 2.4, the process $(g(t, t) \Delta \bar{\mu}_t)_{0 \leq t \leq T}$ is predictable and locally bounded, since $(g(t, t))_{0 \leq t \leq T}$ is continuous and $(\bar{\mu}_t)_{0 \leq t \leq T}$ is locally bounded, being a predictable process of finite variation. Hence, by compensating the process H and using decomposition (2.1), it follows that

$$\begin{aligned} \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) \Delta H_s &= \int_0^t \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) dH_s \\ &= (\text{local martingale})_t + \int_0^t \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) dH_s^p \\ &= (\text{local martingale})_t + \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) \Delta H_s^p. \end{aligned}$$

Recall that, for each $T \in [0, \mathbb{T}]$, the random measures $\mu^{p, Y^{(T)}}$ and $\mu^{p, (Y^{(T)}, H)}$ denote the compensators of $\mu^{Y^{(T)}}$ and $\mu^{(Y^{(T)}, H)}$, respectively, in the sense of [24, Theorem II.1.8]. Hence, by relying on equation (4.12) together with Lemma 2.1, we obtain that

$$\begin{aligned}
 (4.13) \quad & \tilde{X}_t^{(T)} - H_t - [\tilde{X}^{(T)}, H]_t = (\text{local martingale})_t \\
 & + \int_0^t f(s, s)ds - \int_0^t \bar{a}(s, T)ds - \int_0^t \bar{\alpha}(s, T)ds \\
 & + \frac{1}{2} \int_0^t \|\bar{b}(s, T) + \bar{\beta}(s, T)\|^2 ds + \int_0^t g(s, s)m_s ds + \int_0^t g(s, s)d\nu_s \\
 & - (g^{(T)} \star \mu)_t^c - \int_0^t h_s ds - \lambda_t \\
 & - \sum_{0 < s \leq t} \Delta H_s^p + \sum_{0 < s \leq t} (e^{g(s, s)\Delta \bar{\mu}_s} - 1)(1 - \Delta H_s^p) \\
 & + (W^{(1)} \star \mu^{p, Y^{(T)}})_t - (W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t,
 \end{aligned}$$

where we have used the fact that the finite variation process $(g^{(T)} \star \mu)^c$ is predictable, being adapted and continuous. Taking into account equation (4.7) and by [24, Corollary I.3.16], this implies that the discounted defaultable bond price $(P(t, T)/X_t^0)_{0 \leq t \leq T}$ is a local martingale, for every $T \in [0, \mathbb{T}]$, if and only if the predictable finite variation term in (4.13) coincides with $\int_0^t r_s ds$. To this effect, let us suppose that the finite variation terms appearing in (4.13) vanish, for all $0 \leq t \leq T \leq \mathbb{T}$, and analyze separately the absolutely continuous, singular and jump parts. Beginning with the jump parts, it holds that

$$(4.14) \quad -\Delta H_t^p + (e^{g(t, t)\Delta \bar{\mu}_t} - 1)(1 - \Delta H_t^p) + \Delta(W^{(1)} \star \mu^{p, Y^{(T)}})_t - \Delta(W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t = 0.$$

Let $t = T$ and note that, in view of [24, Proposition II.1.17], it holds that

$$\Delta(W^{(1)} \star \mu^{p, Y^{(t)}})_t = \mathbb{E}[\Delta(W^{(1)} \star \mu^{Y^{(t)}})_t | \mathcal{F}_{t-}] = e^{g(t, t)\Delta \bar{\mu}_t} \mathbb{E}[\Delta Y_t^{(t)} | \mathcal{F}_{t-}] = 0,$$

since $\Delta Y_t^{(t)} = 0$, for all $t \in [0, T]$. Similarly, it holds that $\Delta(W^{(1)} \star \mu^{p, (Y^{(t)}, H)})_t = 0$. Therefore,

$$-\Delta H_t^p + (e^{g(t, t)\Delta \bar{\mu}_t} - 1)(1 - \Delta H_t^p) = 0,$$

which corresponds to condition (ii) of Theorem 3.2. In view of (4.14), condition (iii) also follows. Considering now the continuous singular parts of the finite variation terms appearing in (4.13), it must hold that

$$\int_0^t g(s, s)d\nu_s - (g^{(T)} \star \mu)_t^{\text{sing}} - \lambda_t + (W^{(1)} \star \mu^{p, Y^{(T)}})_t^{\text{sing}} - (W^{(2)} \star \mu^{p, (Y^{(T)}, H)})_t^{\text{sing}} = 0,$$

for all $0 \leq t \leq T \leq \mathbb{T}$, which yields condition (v). Finally, considering the densities of the absolutely continuous parts of the finite variation terms appearing in (4.13) and letting $t = T$, it must hold that

$$(4.15) \quad f(t, t) + g(t, t)m_t - (g^{(t)} \star \mu)_t^{\text{ac}} - h_t + (W^{(1)} \star \mu^{p, Y^{(t)}})_t^{\text{ac}} - (W^{(2)} \star \mu^{p, (Y^{(t)}, H)})_t^{\text{ac}} = r_t,$$

for all $t \in [0, \mathbb{T}]$. However, denoting by $A^{(T), \text{ac}}$ the density of the absolutely continuous part of the predictable integrable process $A^{(T)}$ appearing in (3.2), for $T \in [0, \mathbb{T}]$, it holds that

$$(W^{(1)} \star \mu^{p, Y^{(t)}})_t^{\text{ac}} = A_t^{(t), \text{ac}} \int_{\mathbb{R}} (e^{-y} - 1)K^{(t)}(t; dy) = 0,$$

since $K^{(t)}(t; dy) = 0$ for all $t \in [0, \mathbb{T}]$, and it similarly holds that $(W^{(2)} \star \mu^{p, (Y^{(t)}, H)})_t^{\text{ac}} = 0$. Moreover, by the same arguments used in part (d) of the proof of [24, Theorem II.1.8] (but with respect to

the optional σ -field), it can be shown that there exists a kernel $N(\omega, t; du)$ from $(\Omega \times [0, \mathbb{T}], \mathcal{O})$ onto $([0, \mathbb{T}], \mathcal{B}([0, \mathbb{T}]))$ and a predictable integrable increasing process $D = (D_t)_{0 \leq t \leq \mathbb{T}}$ such that $\mu(\omega; ds, du) = N(\omega, t; du) dD_s(\omega)$. Letting $D^c = \int_0^\cdot D_s^{\text{ac}} ds + D^{\text{sing}}$ be the decomposition of the continuous part of the process D into an absolutely continuous part and a singular continuous part, it then follows that, for all $0 \leq t \leq T \leq \mathbb{T}$,

$$(g^{(T)} \star \mu)_t^c = \int_0^t \int_{(s, T]} g(s, u) N(s; du) D_s^{\text{ac}} ds + \int_0^t \int_{(s, T]} g(s, u) N(s; du) dD_s^{\text{sing}},$$

so that $(g^{(t)} \star \mu)_t^{\text{ac}} = 0$, for all $t \in [0, \mathbb{T}]$. Therefore, we have shown that condition (4.15) reduces to

$$f(t, t) + g(t, t)m_t - h_t = r_t,$$

for all $t \in [0, \mathbb{T}]$, which corresponds to condition (i) in Theorem 3.2. Condition (iv) then follows by considering the remaining absolutely continuous terms and making use of condition (i). Conversely, it is easy to see that conditions (i)-(v) of Theorem 3.2 together imply that all the finite variation terms appearing in (4.13) vanish. \square

The following lemma proves relation (3.3).

Lemma 4.3. *Suppose that Assumptions 2.3 and 2.5 hold. Then the compensating measure $\mu^{p, Y^{(T)}}$ is related to the compensating measure μ^p as follows*

$$\int_{\mathbb{R}} y \mu^{p, Y^{(T)}}(\{t\} \times dy) = \int_{(t, T]} g(t, u) \mu^p(\{t\} \times du), \quad 0 \leq t \leq T \leq \mathbb{T}.$$

Proof. It suffices to remark that, in view of [24, §1.11] together with the definition of $Y^{(T)}$ and the predictability of g ,

$$\begin{aligned} \int_{(t, T]} g(t, u) \mu^p(\{t\} \times du) &= \mathbb{E} \left[\int_{(t, T]} g(t, u) \mu(\{t\} \times du) \middle| \mathcal{F}_{t-} \right] = \mathbb{E}[\Delta Y_t^{(T)} | \mathcal{F}_{t-}] \\ &= \mathbb{E} \left[\int_{\mathbb{R}} y \mu^{Y^{(T)}}(\{t\} \times dy) \middle| \mathcal{F}_{t-} \right] = \int_{\mathbb{R}} y \mu^{p, Y^{(T)}}(\{t\} \times dy). \end{aligned}$$

\square

4.3. Proofs of the results stated in Section 3.3.

Proof of Lemma 3.4. By definition of the process $Y^{(T)}$, the random set $\{\Delta Y^{(T)} \neq 0\}$ is a subset of $\{(\omega, t) \in \Omega \times [0, T] : \mu(\omega; \{t\} \times [0, \mathbb{T}]) > 0\}$. Hence, condition (3.7) implies that, up to an evanescent set, $\Delta Y^{(T)} \Delta H = 0$, so that the term $W^{(2)} \star \mu^{(Y^{(T)}, H)}$ is null. \square

Proof of Corollary 3.5. According to [24, Proposition II.1.14], if $\mu(ds, du)$ is an integer-valued random measure on $[0, \mathbb{T}] \times [0, \mathbb{T}]$, there exist a thin random set $D = \bigcup_{n \in \mathbb{N}} \llbracket \sigma_n \rrbracket$ and a $[0, \mathbb{T}]$ -valued optional process γ such that $\mu(\omega; dt, du) = \sum_{s \geq 0} \mathbb{1}_D(\omega, s) \delta_{(s, \gamma_s(\omega))}(dt, dx)$, with $\gamma_s > s$ and where δ denotes the Dirac measure. Observe first that this implies that

$$\bar{\mu}_t = \mu([0, \mathbb{T}] \times [0, t]) = \mu([0, t] \times [0, t]) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\gamma_{\sigma_n} \leq t, \sigma_n \leq t\}} = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\gamma_{\sigma_n} \leq t, \sigma_n < t\}}$$

and, consequently, the decomposition (2.5) reduces to $\bar{\mu}_t = \sum_{0 < s \leq t} \Delta \bar{\mu}_s$. This implies that condition (i) of Theorem 3.2 reduces to condition (i) of the present corollary. Condition (ii) of the corollary is identical to condition (ii) of Theorem 3.2. Note also that

$$(g^{(T)} \star \mu)_t = \sum_{n \in \mathbb{N}} g(\sigma_n, \gamma_{\sigma_n}) \mathbb{1}_{\{\gamma_{\sigma_n} \leq T\}} \mathbb{1}_{\{\sigma_n \leq t\}},$$

so that the continuous part $(g^{(T)} \star \mu)^c$ is null. Moreover, by the definition of the process $Y^{(T)}$,

$$\Delta Y_t^{(T)} = \int_0^T g(t, u) \mu(\{t\} \times du) = \mathbf{1}_{\{t \in D\}} \mathbf{1}_{\{\gamma_t \leq T\}} g(t, \gamma_t),$$

so that

$$\begin{aligned} (W^{(1)} \star \mu^{Y^{(T)}})_t &= \sum_{0 < s \leq t} e^{g(s, s) \Delta \bar{\mu}_s} (e^{-\Delta Y_s^{(T)}} - 1) \\ &= \int_0^t \int_0^T e^{g(s, s) \Delta \bar{\mu}_s} (e^{-g(s, u)} - 1) \mu(ds, du) = (\widehat{W}^{(T)} \star \mu)_t. \end{aligned}$$

Condition (iii) then directly follows from Lemma 3.4, which implies that the term $W^{(2)} \star \mu^{p, (Y^{(T)}, H)}$ appearing in conditions (iii)-(iv)-(v) of Theorem 3.2 is null (up to an evanescent set). Conditions (iv)-(v) of the corollary then immediately follow from conditions (iv)-(v) of Theorem 3.2. \square

Proof of Corollary 3.6. Under the present assumptions, the random measure $\mu(ds, du)$ is an integer-valued random measure. Since $\mathbb{Q}(\tau = S_i) = 0$, for all $i = 1, \dots, N$, condition (3.7) holds, so that the assumptions of Corollary 3.5 are satisfied. The process $(\bar{\mu}_t)_{0 \leq t \leq \mathbb{T}}$ is given by

$$\bar{\mu}_t = \mu([0, \mathbb{T}] \times [0, t]) = \sum_{i=1}^N \mathbf{1}_{\{U_i \leq t\}}, \quad \text{for all } 0 \leq t \leq \mathbb{T},$$

so that conditions (i)-(ii) of the present corollary follow from conditions (i)-(ii) of Corollary 3.5. It remains to compute, for all $0 \leq t \leq T \leq \mathbb{T}$,

$$\begin{aligned} (\widehat{W}^{(T)} \star \mu^p)_t &= \int_0^t \int_{(s, T]} e^{g(s, s) \Delta \bar{\mu}_s} (e^{-g(s, u)} - 1) \mu^p(ds, du) \\ &= \int_0^t \int_{(s, T]} e^{g(s, s) \Delta \bar{\mu}_s} (e^{-g(s, u)} - 1) \xi_s(du) ds \\ &= \int_0^t \int_{(s, T]} (e^{-g(s, u)} - 1) \xi_s(du) ds, \end{aligned}$$

so that condition (iii) of the present corollary follows from condition (iv) of Corollary 3.5. Finally, under the present assumptions, conditions (iii) and (v) of Corollary 3.5 are always satisfied. \square

Proof of Corollary 3.8. Note first that Assumption 2.3 is clearly satisfied under the present assumptions. Moreover, the process $(\bar{\mu}_t)_{0 \leq t \leq \mathbb{T}}$ is simply given by

$$\bar{\mu}_t = \mu([0, \mathbb{T}] \times [0, t]) = \mu(\{0\} \times [0, t]) = \sum_{i=1}^{\infty} \mathbf{1}_{\{u_i \leq t\}}, \quad \text{for all } 0 \leq t \leq \mathbb{T}.$$

Conditions (i)-(ii) then follow from conditions (i)-(ii) of Theorem 3.2. Moreover, for all $0 \leq t \leq T \leq \mathbb{T}$, it holds that

$$Y_t^{(T)} = (g^{(T)} \star \mu)_t = \int_0^T g(0, u) \mu(\{0\} \times du) = \sum_{i=1}^{\infty} g(0, u_i) \mathbf{1}_{\{u_i \leq T\}},$$

so that $Y_t^{(T)} = Y_0^{(T)}$, for all $0 \leq t \leq T \leq \mathbb{T}$. In particular, $\Delta Y^{(T)} = 0$, so that conditions (iii) and (v) of Theorem 3.2 are automatically satisfied, since $\lambda = 0$. Condition (iii) of the corollary then immediately follows from condition (iv) of Theorem 3.2. \square

4.4. Proof of Theorem 3.11.

Proof of Theorem 3.11. In view of Lemma 4.1, credit risky bond prices admit the representation $P(t, T) = \mathcal{E}(R)_t \exp(X_t^{(T)})$, for all $0 \leq t \leq T \leq \mathbb{T}$, where the process $X^{(T)}$ is defined as in (4.6). Hence, by the same arguments of Lemma 4.2, it follows that

$$P(t, T) = \mathcal{E} \left(\tilde{X}^{(T)} + R + [\tilde{X}^{(T)}, R] \right)_t.$$

Note that $[\tilde{X}^{(T)}, R]_t = \sum_{0 < s \leq t} \Delta \tilde{X}_s^{(T)} \Delta R_s$, since R is of finite variation. Moreover, arguing similarly as in (4.9), it holds that

$$\begin{aligned} \Delta \tilde{X}_t^{(T)} (1 + \Delta R_t) &= (e^{\Delta X_t^{(T)}} - 1)(1 + \Delta R_t) \\ &= e^{g(t, t) \Delta \bar{\mu}_t} \left(e^{-\int_t^T g(t, u) \mu(\{t\} \times du)} - 1 \right) + \left(e^{g(t, t) \Delta \bar{\mu}_t} - 1 \right) \left(1 + \int_{[-1, 0]} x \mu^R(\{t\} \times dx) \right) \\ &\quad + e^{g(t, t) \Delta \bar{\mu}_t} \left(e^{-\int_t^T g(t, u) \mu(\{t\} \times du)} - 1 \right) \int_{[-1, 0]} x \mu^R(\{t\} \times dx), \end{aligned}$$

so that

$$\begin{aligned} \sum_{0 < s \leq t} \Delta \tilde{X}_s^{(T)} (1 + \Delta R_s) &= (W^{(1)} \star \mu^{Y^{(T)}})_t + (W^{(3)} \star \mu^{(Y^{(T)}, R)})_t \\ &\quad + \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) \left(1 + \int_{[-1, 0]} x \mu^R(\{s\} \times dx) \right). \end{aligned}$$

Note that all the terms appearing in the last expression are a.s. finite, by the same arguments used after equation (4.11) together with the fact that the process R has bounded jumps. Moreover, it is easy to see that

$$\begin{aligned} \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) \int_{[-1, 0]} x \mu^R(\{s\} \times dx) \\ = (\text{local martingale})_t + \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) \int_{[-1, 0]} x \mu^{p, R}(\{s\} \times dx). \end{aligned}$$

Hence, similarly as in the proof of Theorem 3.2, we obtain that

$$\begin{aligned} \tilde{X}_t^{(T)} + R_t + [\tilde{X}^{(T)}, R]_t &= (\text{local martingale})_t \\ &\quad + \int_0^t f(s, s) ds - \int_0^t \bar{a}(s, T) ds - \int_0^t \bar{\alpha}(s, T) ds + \frac{1}{2} \int_0^t \|\bar{b}(s, T) + \bar{\beta}(s, T)\|^2 ds \\ &\quad + \int_0^t g(s, s) m_s ds + \int_0^t g(s, s) d\nu_s - (g^{(T)} \star \mu)_t^c \\ &\quad - C_t + (W^{(1)} \star \mu^{p, Y^{(T)}})_t + (W^{(3)} \star \mu^{p, (Y^{(T)}, R)})_t \\ &\quad + \sum_{0 < s \leq t} \left(e^{g(s, s) \Delta \bar{\mu}_s} - 1 \right) \left(1 + \int_{[-1, 0]} x \mu^{p, R}(\{s\} \times dx) \right). \end{aligned}$$

Conditions (i)-(v) then follow by a similar analysis as in the proof of Theorem 3.2. \square

5. A STOCHASTIC FUBINI THEOREM WITH RESPECT TO A RANDOM MEASURE

We present here a stochastic Fubini theorem for interchanging the order of integration between an integrable optional random measure and a general, possibly discontinuous, local martingale (this result is used in the proof of Lemma 4.1 in the case where the local martingale is given by a Brownian motion).

Let $X = (X_t)_{t \geq 0}$ be a real-valued local martingale on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{Q})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is assumed to satisfy the usual conditions. Let (E, \mathcal{E}) be a Polish space with its Borel σ -field and $\varrho(\omega; dx)$ an integrable non-negative random measure on E (meaning that $\mathbb{E}[\varrho(E)] < +\infty$). The randomness of the measure $\varrho(\omega; dx)$ prevents the use of classical stochastic Fubini theorems (see, e.g., [34, Theorem IV.64] or [4, Proposition A.2]). Indeed, letting $(H(v, x))_{v \geq 0}$ be a predictable process, measurable in x and integrable with respect to the local martingale X , for every $x \in E$, a naive interchange of the order of integration would yield

$$(5.1) \quad \int_E \int_0^t H(v, x) dX_v \varrho(dx) = \int_0^t \int_E H(v, x) \varrho(dx) dX_v.$$

However, on the right-hand side of the above equality, one ends up with a stochastic integral where the integrand $\int_E H(v, x) \varrho(dx)$ typically fails to be adapted to the filtration \mathbb{F} .

We now derive a sufficient condition for making possible the change of the order of integration in (5.1). The key idea (compare with part (iii) of Assumption 2.3) consists in assuming that adding the full knowledge of the random measure $\varrho(\omega; dx)$ to the initial σ -field \mathcal{F}_0 does not destroy the semimartingale property of X (but may destroy the local martingale property of X) and then performing a change of the order of integration in an enlarged filtration, to which the integrand appearing in the right-hand side of (5.1) will be adapted. The proof will use elements from the theory of enlargement of filtrations, together with the theory on stochastic integration depending on a parameter developed in [39].

To this end, let us define the enlarged filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$ by $\tilde{\mathcal{F}}_t := \bigcap_{s > t} \tilde{\mathcal{F}}_s^0$, where

$$\tilde{\mathcal{F}}_t^0 := \mathcal{F}_t \vee \sigma\{\varrho(A); A \in \mathcal{E}\}, \quad \text{for all } t \geq 0.$$

The filtration $\tilde{\mathbb{F}}$ corresponds to the original filtration \mathbb{F} *initially enlarged* with respect to the random measure $\varrho(\omega; dx)$. In particular, note that $\varrho(\omega; dx)$ is measurable with respect to the initial σ -field $\tilde{\mathcal{F}}_0^0$ of the enlarged filtration. This property allows establishing the following proposition, whose proof follows similar arguments to [34, Theorems IV.64-65]. We denote by $L_m(X; \mathbb{F})$ ($L(X; \tilde{\mathbb{F}})$, resp.) the set of all \mathbb{F} -predictable ($\tilde{\mathbb{F}}$ -predictable, resp.) real-valued processes which are integrable in the local martingale sense (in the semimartingale sense, resp.) with respect X in the filtration \mathbb{F} ($\tilde{\mathbb{F}}$, resp.).

Proposition 5.1. *Let $H : \Omega \times \mathbb{R}_+ \times E \rightarrow \mathbb{R}$ be a $\mathcal{P} \otimes \mathcal{E}$ -measurable function such that $(H_t(x))_{t \geq 0} \in L_m(X; \mathbb{F})$, for all $x \in E$, and let $Z : \Omega \times \mathbb{R}_+ \times E \rightarrow \mathbb{R}$ be an $\mathcal{O} \otimes \mathcal{E}$ -measurable function such that, for all $x \in E$, the process $(Z(t, x))_{t \geq 0}$ is indistinguishable from $(\int_0^t H_v(x) dX_v)_{t \geq 0}$. Suppose that the \mathbb{F} -local martingale X is a semimartingale in the enlarged filtration $\tilde{\mathbb{F}}$ and that*

$$(5.2) \quad \left(\sqrt{\int_E (H_t(x))^2 \varrho(dx)} \right)_{t \geq 0} \in L(X; \tilde{\mathbb{F}}).$$

Then the process $(\int_E Z(t, x) \varrho(dx))_{t \geq 0}$ is a càdlàg version of the process $(\int_0^t \hat{H}_v dX_v)_{t \geq 0}$, where $\hat{H}_v := \int_E H_v(x) \varrho(dx)$ and where the stochastic integral $\int_0^t \hat{H}_v dX_v$ has to be understood as a semimartingale stochastic integral in the enlarged filtration $\tilde{\mathbb{F}}$.

Proof. Note first that, since $H(x) \in L_m(X; \mathbb{F})$, for all $x \in E$, [39, Theorem 2 and note on page 133] gives the existence of an $\mathcal{O} \otimes \mathcal{E}$ -measurable function $Z : \Omega \times \mathbb{R}_+ \times E \rightarrow \mathbb{R}$ such that, for all $x \in E$, the process $(Z(t, x))_{t \geq 0}$ is indistinguishable from $(\int_0^t H_v(x) dX_v)_{t \geq 0}$.

Similarly as in the proof of [34, Theorem IV.64], by (pre-)stopping we may assume without loss of generality that X is a bounded martingale in \mathbb{F} . Suppose first that the process $(H_t(x))_{t \geq 0}$ is

of the form $H_t(\omega; x) = K_t(\omega)h(x)$, for all $(t, x) \in \mathbb{R}_+ \times E$, where $K = (K_t)_{t \geq 0}$ is an \mathbb{F} -predictable bounded process and $h : E \rightarrow \mathbb{R}$ is a bounded measurable function. Clearly, in this case it holds that $Z(t, x) = h(x) \int_0^t K_v dX_v$. Moreover, up to an evanescent set, it holds that

$$\begin{aligned} \int_E Z(t, x) \varrho(dx) &= \int_E h(x) \int_0^t K_v dX_v \varrho(dx) = \int_0^t K_v dX_v \int_E h(x) \varrho(dx) \\ (5.3) \quad &= \int_0^t \left(\int_E h(x) \varrho(dx) K_v \right) dX_v = \int_0^t \hat{H}_v dX_v. \end{aligned}$$

Indeed, under the present assumptions, the stochastic integral $\int_0^\cdot K_v dX_v$ can be understood both in the filtration \mathbb{F} as well as in the enlarged filtration $\tilde{\mathbb{F}}$ (since the integrand K is bounded and \mathbb{F} -predictable and, hence, belongs to $L_m(X; \mathbb{F}) \cap L(X; \tilde{\mathbb{F}})$; compare with [24, Proposition III.6.25]) and where the stochastic integral $\int_0^\cdot \hat{H}_v dX_v$ is to be understood in $\tilde{\mathbb{F}}$. Note that the third equality in (5.3) uses the fact that the stochastic integral $\int_0^\cdot K_v dX_v$ is well-defined as a semimartingale stochastic integral in the enlarged filtration $\tilde{\mathbb{F}}$ and the random variable $\int_E h(x) \varrho(dx)$ is $\tilde{\mathcal{F}}_0$ -measurable. By linearity, (5.3) also holds for the vector space \mathbb{V} of all finite linear combinations of processes of the form $H_t(\omega; x) = K_t(\omega)h(x)$ as above.

By a monotone class argument, the claim then holds for all bounded $\mathcal{P} \otimes \mathcal{E}$ -measurable functions $H : \Omega \times \mathbb{R}_+ \times E \rightarrow \mathbb{R}$. Indeed, it suffices to show that if $\{H^n\}_{n \in \mathbb{N}} \subseteq \mathbb{V}$ and $\lim_{n \rightarrow +\infty} H^n = H$, with H $\mathcal{P} \otimes \mathcal{E}$ -measurable and bounded, then the claim holds for H . To this effect, letting $Z^n(\cdot, x)$ be the $\mathcal{O} \otimes \mathcal{E}$ -measurable version of $\int_0^\cdot H_v^n(x) dX_v$, we compute, for an arbitrary $T \geq 0$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_E Z^n(t, x) \varrho(dx) - \int_E Z(t, x) \varrho(dx) \right| \right] \leq \mathbb{E} \left[\int_E \sup_{t \in [0, T]} |Z^n(t, x) - Z(t, x)| \varrho(dx) \right]$$

and, using the Jensen and Cauchy-Schwarz inequalities, conditioning and the Fubini-Tonelli theorem, we obtain

$$\begin{aligned} \left(\mathbb{E} \left[\int_E \sup_{t \in [0, T]} |Z^n(t, x) - Z(t, x)| \varrho(dx) \right] \right)^2 &\leq \mathbb{E} \left[\varrho(E) \int_E \sup_{t \in [0, T]} |Z^n(t, x) - Z(t, x)|^2 \varrho(dx) \right] \\ &= \mathbb{E} \left[\varrho(E) \int_E \mathbb{E} \left[\sup_{t \in [0, T]} |Z^n(t, x) - Z(t, x)|^2 \middle| \tilde{\mathcal{F}}_0 \right] \varrho(dx) \right], \end{aligned}$$

where we take an \mathcal{E} -measurable version of the $\tilde{\mathcal{F}}_0$ -conditional expectation, which exists due to [39, Lemma 3]. By assumption, X is a $\tilde{\mathbb{F}}$ -semimartingale. Being bounded, it admits a canonical decomposition in the enlarged filtration $\tilde{\mathbb{F}}$ of the form $X = M + B$, where M is an $\tilde{\mathbb{F}}$ -local martingale and B is an $\tilde{\mathbb{F}}$ -predictable process of finite variation. Recalling that a bounded predictable process belongs is integrable with respect to both M and B in $\tilde{\mathbb{F}}$, we can then write, using conditional Doob's inequality,

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |Z^n(t, x) - Z(t, x)|^2 \middle| \tilde{\mathcal{F}}_0 \right] &= \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t (H_s^n(x) - H_s(x)) (dM_s + dB_s) \right|^2 \middle| \tilde{\mathcal{F}}_0 \right] \\ &\leq 2 \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t (H_s^n(x) - H_s(x)) dM_s \right|^2 \middle| \tilde{\mathcal{F}}_0 \right] + 2 \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t (H_s^n(x) - H_s(x)) dB_s \right|^2 \middle| \tilde{\mathcal{F}}_0 \right] \\ &\leq 8 \mathbb{E} \left[\left(\int_0^T (H_s^n(x) - H_s(x)) dM_s \right)^2 \middle| \tilde{\mathcal{F}}_0 \right] + 2 \mathbb{E} \left[\left(\int_0^T |H_s^n(x) - H_s(x)| d\text{Var}(B)_s \right)^2 \middle| \tilde{\mathcal{F}}_0 \right] \end{aligned}$$

$$\leq 8 \mathbb{E} \left[\int_0^T (H_s^n(x) - H_s(x))^2 d[M]_s \Big| \widetilde{\mathcal{F}}_0 \right] + 2 \mathbb{E} \left[\left(\int_0^T |H_s^n(x) - H_s(x)| d\text{Var}(B)_s \right)^2 \Big| \widetilde{\mathcal{F}}_0 \right],$$

where $\text{Var}(B)$ denotes the total variation process of B and the two integrals $\int_0^T (H_v^n(x) - H_v(x))^2 d[M]_v$ and $\int_0^T |H_s^n(x) - H_s(x)| d\text{Var}(B)_s$ admit versions which are \mathcal{E} -measurable by [39, Lemma 2]. Hence,

$$\begin{aligned} \left(\mathbb{E} \left[\int_E \sup_{t \in [0, T]} |Z^n(t, x) - Z(t, x)| \varrho(dx) \right] \right)^2 &\leq 8 \mathbb{E} \left[\varrho(E) \int_E \int_0^T (H_s^n(x) - H_s(x))^2 d[M]_s \varrho(dx) \right] \\ &\quad + 2 \mathbb{E} \left[\varrho(E) \int_E \left(\int_0^T |H_s^n(x) - H_s(x)| d\text{Var}(B)_s \right)^2 \varrho(dx) \right] \end{aligned}$$

Since $|\Delta M| \leq |\Delta X| + \rho |\Delta X|$ (compare with [24, Lemma I.4.24]) and X is assumed to be bounded, we can further suppose by localization that $[M]_T + \text{Var}(B)_T \leq K$ a.s., for some constant $K > 0$. By arguing as in [34, proof of Theorem IV.64], we can let $n \rightarrow +\infty$ and apply the dominated convergence theorem, thus yielding that the process $\int_0^T \hat{H}_v^n dX_v = \int_E Z^n(\cdot, x) \varrho(dx)$ converges uniformly on compacts in probability to $\int_E Z(\cdot, x) \varrho(dx)$. Since $\int_0^T \hat{H}_v^n dX_v$ converges uniformly on compacts in probability to $\int_0^T \hat{H}_v dX_v$ by the dominated convergence theorem for stochastic integrals (see, e.g., [34, Theorem IV.32]), with both stochastic integrals being understood in the enlarged filtration $\widetilde{\mathbb{F}}$, it follows that $\int_0^T \hat{H}_v dX_v$ is a version of $\int_E Z(t, x) \varrho(dx)$. Since $T \geq 0$ is arbitrary, we have thus established the proposition in the case where H is a bounded $\mathcal{P} \otimes \mathcal{E}$ -measurable function.

Since by assumption $H(x) \in L_m(X; \mathbb{F})$, for all $x \in E$ (and, hence, by localization, we can assume that the stochastic integral $\int_0^T H_v(x) dX_v$ is a bounded martingale in \mathbb{F}), the result can then be extended to all $\mathcal{P} \otimes \mathcal{E}$ -measurable functions satisfying condition (5.2) by considering the truncated functions $H_v^n(x) := H_v(x) \mathbf{1}_{\{|H_v(x)| \leq n\}}$, for each $n \in \mathbb{N}$, and proceeding exactly as in [34, proof of Theorem IV.65]. \square

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LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES, UNIVERSITÉ PARIS DIDEROT, AVENUE DE FRANCE, 75205, PARIS, FRANCE.

E-mail address: `fontana@math.univ-paris-diderot.fr`

FREIBURG UNIVERSITY, DEP. OF MATHEMATICS, ECKERSTR. 1, 79104 FREIBURG, GERMANY.

E-mail address: `thorsten.schmidt@stochastik.uni-freiburg.de`